RUNGE-KUTTA METHODS FOR NUMERICAL INTEGRATION OF EQUATIONS OF MOTION FOR GLONASS SATELLITES

Abstract. Several types of methods can solve equations of satellite motion numerically. These methods are divided into single and multi-step methods. The accuracy of each method depends directly on adopted integration step size between successive iterations. To achieve result with required accuracy it is important to maintain appropriate size of integration step. Inappropriate step size could cause local errors between iterations greater than accuracy of the method. Therefore, integration step size needs to be reduced until it does not affect accuracy of the final solution. Group of Runge-Kutta (RK) methods for solving equations of satellite motion have been analysed in this article. Five different methods: Runge-Kutta 4th order, Runge-Kutta 5th order and Runge-Kutta-Fehlberg 4th and 5th order methods were discussed. Compared to the classical Runge-Kutta integration method other methods are slower, but give results that are slightly more accurate.

KEYWORDS**:** numerical integration, GNSS, GLONASS, satellite geodesy

1 INTRODUCTION

Equations of satellite motion could be solved both analytically (Góral & Skorupa, 2012) and numerally (Gaglione et al., 2011). Runge-Kutta (RK) methods are one of the well-known numerical methods for solving differential equations (Kosti, Anastassi, & Simos, 2009; Ozawa, 1999; Sermutlu, 2004), while 4th order Runge-Kutta method is recommended to solve equations of satellite motion by GLONASS Interface Control Document (ICD-GLONASS, 2008).

Currently there are very few publications concerning comparison of numerical methods for solving GNSS equations of satellite motion. Numerical integration of low Earth orbiting satellites was performed by (Es-hagh, 2005). Author compared two variable step integration methods: Adams and Runge-Kutta-Fehlberg (RKF) methods. Adams' method is recommended for a long arc orbit integration or in low resolutions (large step size) orbit integration. In contrast, RKF method is recommended to be used for high-resolution (small step size) solutions. (Sermutlu, 2004) presented accuracy and speed comparison tests of Runge-Kutta 4th and 5th order for solving Lorenz equation. He noticed that 4th order method gives more accurate results for shorter running times, but as step sizes decline 5th order method gives more accurate results. The same results were obtained by (Khodabin & Rostami, 2015), the authors analysed different orders of Runge-Kutta methods for applications in electric circuits. The authors confirmed superiority of higher order RK methods over other. (O. Montenbruck, 1992) compares multistep, interpolation and Runge-Kutta methods for the numerical integration of ordinary differential equations of orbital motion. Author showed that both single-step and multi-step methods are competitive. Equations of satellite motion was also solved by many different approaches, e.g. Runge-Kutta-Fehlberg method (Atanassov, 2010), analytically (Kudryavtsev, 1995), by MATLAB ODE45 function (Bradley, Jones, Beylkin, Sandberg, & Axelrad, 2014) or by new types of Runge-Kutta methods (Gonzalez, Pablo, & Lopez, 1999).

Runge-Kutta 4th order method to solve equations of satellite motion was presented by (ICD-GLONASS, 2008). He proved that this kind of solution gives satisfactory results - single GLONASS satellite position is determined with 1-2 m accuracy. It is clearly seen that the error in orbit integration strongly depends on a step size. GLONASS satellite integration results have no explicit differences between solutions from 1 to 300 s integration step size. The author suggested that 60 s GLONASS integration step width is sufficient in any case, due to the fact that for small angular distances the satellite orbit could be considered as nearly linear.

2 KEPLERIAN MOTION

Simplified satellite orbiting is called Keplerian motion (Zare, 1982). In Earth-artificial satellite setting the mass of a satellite can be considered negligible and does not enter the motion equations system (Breiter & Elipe, 2006). This is due to its size and mass that are negligibly small relatively to the mass of the Earth. The satellites motion is govern by the Newton's second law hence it is expressed by :

$$\ddot{\vec{\mathbf{r}}} = -\frac{\mu}{r^2}\frac{\mathbf{r}}{r} \tag{1}$$

where:

$\mu = GM$ - the product of Newton's gravitational constant and mass of the Earth

r - distance between the Earth and satellite centres

Equation 1 relates to a motion in an inertial system. Two vectors or 6 scalars are the solutions of this second order differential equation (Keplerian elements). They are the results of double integration of (1). In case of the Earth's artificial satellite, perturbing forces affecting its position should also be taken into account (Orbit & Perturbations, 2011):

$$\ddot{\vec{\mathbf{r}}} = -\frac{\mu}{r^2}\frac{\mathbf{r}}{r} + \mathbf{K} \tag{2}$$

where:

K - perturbing forces

Gravitational forces due to the Earth as well as the strength of perturbing forces determine satellites motion. Table 1 shows the magnitude of perturbing forces and their effect on a GNSS satellite.

Source	*Acceleration [m/s2]*	*Orbit error after 24 hours [m]*
Two-body term of the Earth's gravity field	0.59	∞
Oblateness of the Earth	$5 \cdot 10^{-5}$	10 000
Lunar gravitational attraction	$5 \cdot 10^{-6}$	3 000
Solar gravitational attraction	$2 \cdot 10^{-6}$	800
Other terms of the Earth's gravity field	$3 \cdot 10^{-7}$	200
Radiation pressure (direct)	$9 \cdot 10^{-8}$	200
Y-bias	$5 \cdot 10^{-10}$	2
Solid Earth tides	$1 \cdot 10^{-9}$	0.3

Table 1: Perturbing accelerations acting on GPS satellite (Dach, Hugentobler, Fridez, & Meindl, 2007).

The main perturbing force affecting a satellite is the Earth's oblateness that characterizes polar flattening of the Earth. The effect of accelerations due to luni-solar gravitational perturbations is an order of magnitude smaller than the second zonal harmonic. Other forces can be considered as negligible. It may be assumed that perturbing forces acting on a GPS satellite affect will be different that on a GLONASS satellites due to two reasons. Firstly GLONASS satellite orbit the Earth much lower, that is mean they are much sensitive to gravitational perturbations. Secondly GLONASS satellites have larger area-to-mass ratios than GPS satellites, which implies that the impact of solar radiation pressure is larger for GLONASS.

3 RUNGE-KUTTA METHODS

Numerical integration methods could be classified into single and multi-step methods. In case of multi-step methods to calculate the future value of the function, values of the function at some previous time points (e.g. t_{n-1}, t_{n-2}) must be known. The best known multi-step methods used to solve equations of satellite motion is Cowell and Encke methods (Liu & Liao, 1994). Whereas single-step methods basing on single, initial point of time calculate future values of the function. The best-known single-step methods for solving satellite equations of motion are Runge-Kutta 4th and higher order methods.

The equation of satellite's motion is a second order differential equation. Therefore it has to be converted to system of first order differential equation to be solved by RK methods as following:

$$
\begin{aligned}
y'(x) &= f(x, y(x)) \\
y(t_0) &= y_0
\end{aligned}
\tag{3}
$$

Runge-Kutta method allows calculation of the approximate value of the function $y(x_n)$ for $a = x_0 < x_1 < \ldots < x_n = b$, as formula:

$$
\begin{aligned}
y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i k_i \\
x_{n+1} &= x_n + h
\end{aligned}
\tag{4}
$$

where:

$$
k_i = f\left(x_n + c_i h,\ y_n + h\sum_{j=1}^{i-1} a_{ij} k_j\right)
\tag{5}
$$

$i=2,3,\ldots,s$

and

$$
c_i = \sum_{j=1}^{i-1} a_{ij}
\tag{6}
$$

where:

a, b - constants

h - step size

s - Runge-Kutta method's order

$i=2,3,\ldots,s$

Expanding (2) into first order differential equations still makes it impossible to solve them analytically in a fast and simple way. GLONASS Interface Control Document (ICD-GLONASS, 2008) recommends the use Runge-Kutta 4th order method for this purpose, as it ensures adequate accuracy altogether with the simplicity of the solution. Equation (7) is an extension of (2) into a form of scalar functions. It takes into account perturbing forces due to the flattening of the Earth (second zonal harmonic) and influence of the Sun and Moon (Poutanen, Vermeer, & Jaakko, 1996):

$$
\begin{aligned}
&\frac{dx}{dt} = \dot{x} \\
&\frac{dy}{dt} = \dot{y} \\
&\frac{dz}{dt} = \dot{z} \\
&\frac{d\dot{x}}{dt} = -\frac{\mu}{r^3}x + \frac{3}{2}C_{20}\frac{\mu a^2}{r^5}x\left(1 - \frac{5z^2}{r^2}\right) + \ddot{x}_{LS} + \omega^2 x + 2\omega y \\
&\frac{d\dot{y}}{dt} = -\frac{\mu}{r^3}y + \frac{3}{2}C_{20}\frac{\mu a^2}{r^5}y\left(1 - \frac{5z^2}{r^2}\right) + \ddot{y}_{LS} + \omega^2 y - 2\omega x \\
&\frac{d\dot{z}}{dt} = -\frac{\mu}{r^3}z + \frac{3}{2}C_{20}\frac{\mu a^2}{r^5}z\left(3 - \frac{5z^2}{r^2}\right) + \ddot{z}_{LS}
\end{aligned}
\tag{7}
$$

where:

x, y, z - satellite coordinates

$\dot{x}, \dot{y}, \dot{z}$ - satellite velocities

$\ddot{x}_{LS}, \ddot{y}_{LS}, \ddot{z}_{LS}$ - lunisolar accelerations

a - semi-major axis of the ellipsoid

ω - the Earth rotation rate

C_{20} - second zonal harmonic coefficient of the geopotential

$r = \sqrt{x^2 + y^2 + z^2}$

μ = GM

Second zonal harmonic is known from parameters of current PZ-90 realization. In calculations it is adopted as the known parameter. Lunisolar accelerations are varying in time thus they are transmitted in GLONASS navigational (broadcast) message in 15 min intervals, and they are assumed constant within ±15 min around initial position.

4 GLONASS NAVIGATION MESSAGE

GLONASS navigation message contains information regarding satellites' position parameters for a single observation epoch. These data are recorded in RINEX format (Gurtner & Estey, 2007) with 30-minutes interval as vector components of satellite position, velocity and acceleration (Table 2).

Observation record	*Description*	*Format*
SV / EPOCH / SV CLK	- Satellite system (R), satellite number (slot number in sat. constellation) - Epoch: Toc - Time of Clock (UTC) __- year (4 digits) __- month, day, hour, minute, second - SV clock bias (sec) (-TauN) - SV relative frequency bias (+GammaN) - Message frame time (tk+nd*86400) _ in seconds of the UTC week	A1, I2.2 1X, I4 5 (1X, I2, 2), 3D19.12
BROADCAST ORBIT – 1	- Satellite position X __ __ __ (km) - Satellite velocity X dot__ __ (km/sec) - Satellite X acceleration__ __ (km/sec2) - Satellite health (0=OK)	4X, 4D19.12
BROADCAST ORBIT – 2	- Satellite position Y________(km) - Satellite velocity Y dot__ __ (km/sec) - Satellite Y acceleration__ __ (km/sec2) - Satellite frequency number__(-7…+12)	4X, 4D19.12
BROADCAST ORBIT – 3	- Satellite position Z________ (km) - Satellite velocity Z dot__ __ (km/sec) - Satellite Z acceleration__ __ (km/sec2) - Age of oper. information__ _(days)	4X, 4D19.12

Table 2: GLONASS data record description (Gurtner & Estey, 2007).

Figure 1 contains a single record of GLONASS navigational message in RINEX format. It relates to satellite PNR 1 from 9th June 2013 0:00 GLONASS time.

```
PRN y m d h m s          SV clock bias (sec) SV relative frequency bias Message frame time
1 13 6 9 0 0 0.0         -0.172111205757E-03 0.000000000000E+00         0.846000000000E+05
Satellite   position    XX velocity (km/sec) X acceleration (km/sec2)   Health
(km)
0.144409179688E+05       -0.264622497559E+01 0.000000000000E+00         0.000000000000E+00
Satellite   position    YY velocity (km/sec) Y acceleration (km/sec2)   Frequency number
(km)
0.522635791016E+04       0.877996444702E+00  0.000000000000E+00         0.100000000000E+01
Satellite   position    ZZ velocity (km/sec) Z acceleration (km/sec2)   Age of oper.
(km)                                                                    information
0.203675307617E+05       0.165256118774E+01  -0.279396772385E-08        0.000000000000E+00
```

Figure 1: Example of GLONASS navigation message

Contrary to GPS, GLONASS message contains information about satellites positions in ECEF coordinate system (Gaglione et al., 2011). These data for a single satellite are stored in four 80-byte lines (Figure 1). GLONASS ephemeris message contains information about satellites' position in current PZ-90 realization (Boucher & Altamimi, 2001). PZ-90.02 realization was obligatory since 2007 (Oliver Montenbruck, Steigenberger, & Hauschild, 2015), currently PZ-90.11 is use (IGSMAIL-6896).

5 GENERAL COMPARISON

In this paper group of Runge-Kutta methods were analysed in resolving equations of satellite motion for GLONASS satellites. Parameters of GLONASS space segment are presented in Table 3:

Parameter	*Value*
Number of SV	24
Orbital planes	3
Orbital altitude (km)	19 100
Orbital inclination	64.8°
Ground track period	8 sidereal days
Layout	Symmetric
Broadcast ephemerides	ECEF
Datum	PZ-90

Table 3: GLONASS space segment parameters (Angrisano, Gaglione, & Gioia, 2013).

This paper discusses four variants of Runge-Kutta method: best-known 4th order method (RK4), 5th order method (RK5) and Runge-Kutta-Fehlberg 4th (RKF4) and 5th (RKF5) order methods. Table 4 shows formulas of analysed RK methods:

Runge-Kutta 4th order (RK4)	*Runge-Kutta 5th order (RK5)*	*Runge-Kutta-Fehlberg (RKF45)*
$k_1 = hf(t_n, y_n)$ $k_2 = hf\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$ $k_3 = hf\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$ $k_4 = hf(t_n + h, y_n + k_3)$ $y_{n+1} = y_n + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$	$k_1 = hf(t_n, y_n)$ $k_2 = hf\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$ $k_3 = hf\left(t_n + \frac{1}{4}h, y_n + \frac{3k_1 + k_2}{16}\right)$ $k_4 = hf\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_3\right)$ $k_5 = hf\left(t_n + \frac{3}{4}h, y_n + \frac{-3k_2 + 6k_3 + 9k_4}{16}\right)$ $k_6 = hf\left(t_n + h, y_n + \frac{k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5}{7}\right)$ $y_{n+1} = y_n + \frac{7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6}{90}$	$k_1 = hf(t_n, y_n)$ $k_2 = hf\left(t_n + \frac{1}{4}h, y_n + \frac{1}{4}k_1\right)$ $k_3 = hf\left(t_n + \frac{3}{8}h, y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2\right)$ $k_4 = hf\left(t_n + \frac{12}{13}h, y_n + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right)$ $k_5 = hf\left(t_n + h, y_n + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right)$ $k_6 = hf\left(t_n + \frac{h}{2}, y_n - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 + \frac{11}{40}k_5\right)$ $y_{n+1}^{[4]} = y_n + \left(\frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5\right)$ $y_{n+1}^{[5]} = y_n + \left(\frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6\right)$

Table 4: Parameter of analysed Runge-Kutta methods.

Satellite position's error depends on Runge-Kutta method order and adopted for calculations integration step. In principle, position determination is more accurate for smaller integration steps. Smaller integration step carries a serious increase of intermediate positions thus, increasing computation time. Each step *h,* depending on the adopted formula (Table 4) requires calculation of four, five or six intermediate values of the function. Therefore the best solution appears to be a method which provides required accuracy of satellite position solution combined with the highest execution speed. It is especially important in case of real-time solutions.

In this paper the position of #10 GLONASS satellite ((SV 717, orbit 2, launched 25/12/2006, active from 03/04/2007)) due date 01/01/2012 has been analysed in three moments of time: 10^{15}, 10^{45} and 11^{15} UTC. Comparison of numerical solutions of (2) was carried out on the basis of author's own scripts implemented in Matlab R2010b®. They were run on Lenovo L420 computer equipped with Windows 7 Professional, with the Intel Core i5-2410M 2.30 GHz, 4.00 GB RAM.

6. RESULTS

This paper shows research of GLONASS' satellite position determination by RK methods according to integration step size and its effect on the accuracy and speed of solution. Survey is based on broadcast orbit coordinates. The determination of position accuracy was obtained by comparison with RNXPRE Bernese GPS Software 5.0 (Dach et al., 2007) results, which were adopted as a model.

Based on known initial function values of position, velocity and acceleration it is possible to determine satellite's position for any moment within range ±15 min (900 s). This time span comes from the fact that the GLONASS ephemeris is updated every 30 minutes. If ephemeris data is used in the range exceeding ±15min difference between calculated and actual position expected is grow rapidly every ±15 min (Figure 2).

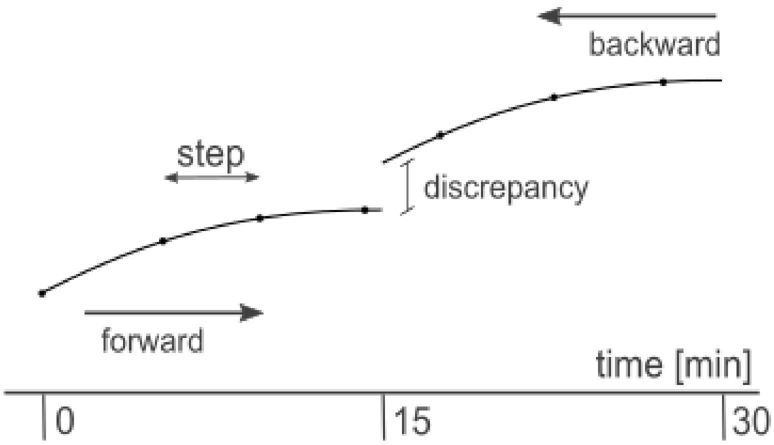

Figure 2: Discrepancy of forward and backward 15 - minute integration

Figure 3 shows errors of XYZ components calculated based on initial satellite position by RK4 with integration step h = 30 s. After 30 minutes, the error of each component does not exceed 1 meter, after 60 minutes error is smaller than 5 meters, and after 4 hours exceeds value of several meters. Therefore in an application of numerical methods for solving equations of satellite motion information on satellite position in the shortest possible time intervals is very important.

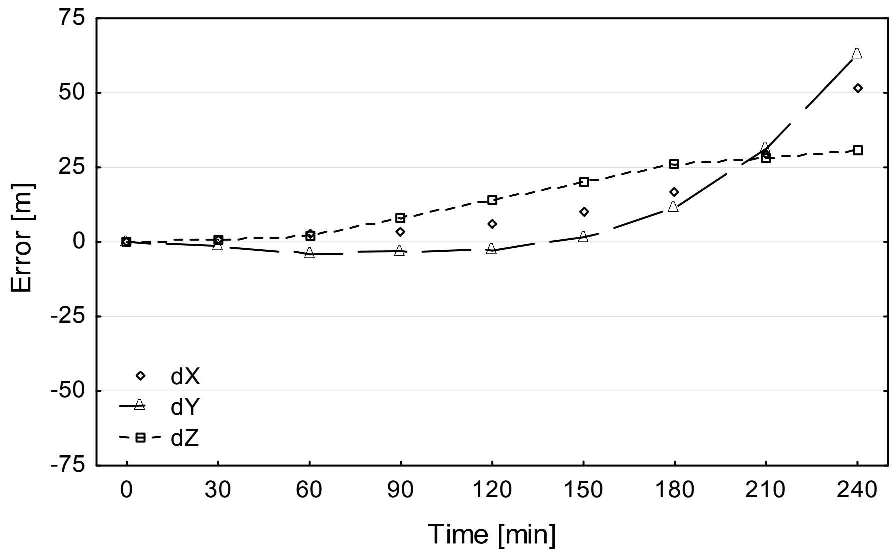

Figure 3: Increase of satellite position error (RK4, h = 30 s)

Figure 4 shows more detailed data presented on Figure 3. "Known" coordinate and speed components are at $t = 0$ s. At $t = 900$ s follows update of satellite ephemeris data and then should be used next "known" position coordinate (t = 1800 s for this figure) and solved backward. So increase of XYZ components error magnitude due to updated ephemeris parameters is very clearly visible.

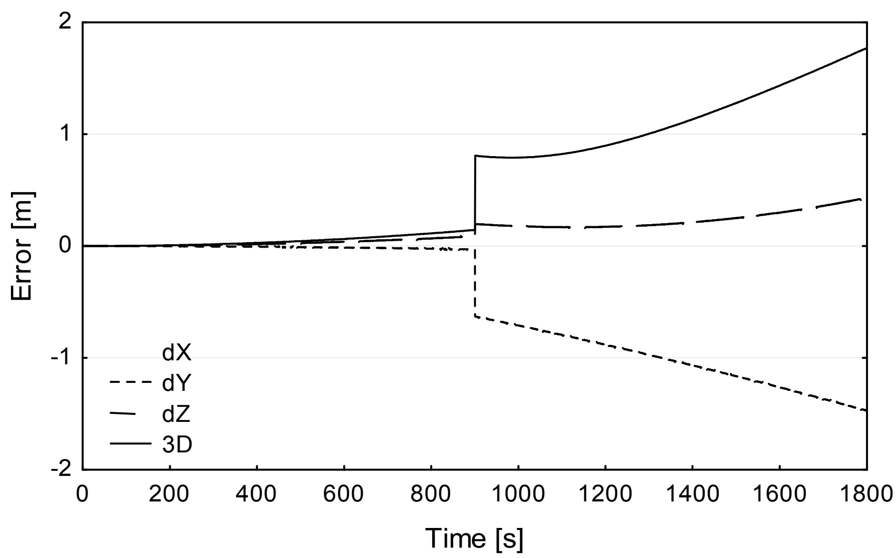

Figure 4: Increase of satellite position error (RK4, h = 1 s)

Figure 5 shows the difference between RK4 method with step h = 1 s solution and the reference solution. The figure presents three consecutive "backward" and "forward" solutions within 900 s interval. At 900 s, 2700 s and 4500 s moments satellite coordinates, velocity and acceleration values are known. Solutions of three analysed, successive time points have similar errors. The offset of each component is a result of its update. That is why determination of single satellite position should be done within ±900 s around known position. Component X maximum error is around -0.1 m, Y around 0.9 m, and Z component up to 0.1 m error. Consequently, maximum

3D position error is 0.15 m. Thus, this type of calculation can be considered as sufficient for GLONASS broadcast orbit determination, due to its accuracy of about several meters.

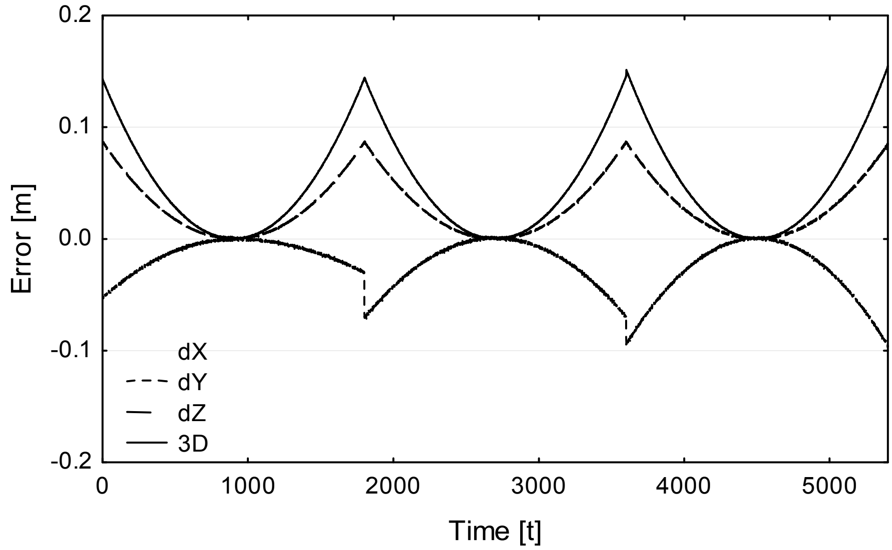

Figure 5: Example of three consecutive integration steps

Table 5 presents a comparison of average speed of satellite position determination. These values are means of 100 000 consecutive solutions of Runge-Kutta methods. It depends on adopted integration step size *h*. Increased integration step size decreases time of position determination. For each integration step the most efficient is Runge-Kutta 4th order method (RK4), due to the least complexity. The other three methods depending on the step length are between 2 to 6 times slower than RK4 method. Despite of the most complex equations RKF5 method is the second fastest after the RK4 method among analysed. RKM is the slowest method for each step size. Speed of calculation in this method is comparable to other only for 1 and 3 s integration step sizes.

Step size h [s]	*1*	*3*	*5*	*10*	*20*	*30*	*90*	*180*	*300*	*900*
Number of steps	*900*	*300*	*180*	*90*	*45*	*30*	*10*	*5*	*3*	*1*
RK4	4.816	1.605	0.962	0.480	0.241	0.160	0.054	0.027	0.016	0.006
	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
RK5	7.374	2.881	1.758	0.992	0.468	0.353	0.147	0.084	0.060	0.038
	153%	180%	183%	207%	194%	221%	272%	311%	375%	633%
RKF4	7.712	2.539	1.557	0.776	0.401	0.275	0.107	0.066	0.049	0.031
	160%	158%	162%	162%	166%	172%	198%	244%	306%	517%
RKF5	8.047	2.719	1.608	0.774	0.403	0.275	0.113	0.079	0.054	0.033
	167%	169%	167%	161%	167%	172%	209%	293%	338%	550%

Table 5: Average duration of positions calculation and percentage changes in relation to RK4 [ms].

Figure 6 presents calculated errors of "forward" satellite position. It reveals the difference between the author's and model solution based on integration step size. In case of small step length, less than 180 s, results are comparable for all tested methods and the maximum error does not exceed 0.15 m. This accuracy is sufficient for navigation purposes.

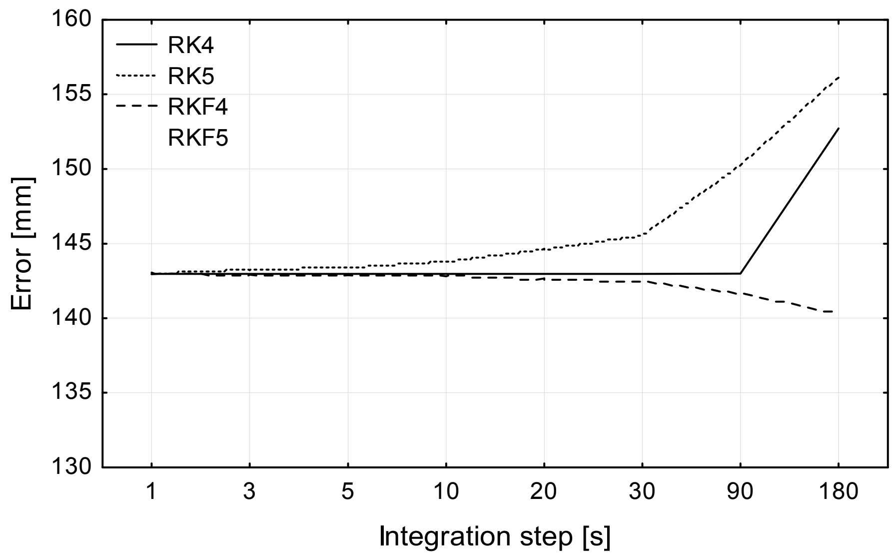

Figure 6: Runge-Kutta method determination error [mm]

With the increase of integration step length a distinct advantage of higher order Runge-Kutta methods may be observed. It is clearly visible for integration steps h = 300 s and h = 900 s. RKF method projects satellite's trajectory with 0.60 cm accuracy for a single, 900 s step.

7 CONCLUSIONS

The accuracy of GLONASS satellite's position calculated numerically depends mostly on integration step size. The influence of applied RK method type and order is smaller. Short integration step allows a relatively high precision, but it involves extension of solution time. Error of calculated position from initial parameter (epoch) increases together with "distance" from known coordinates. This study confirmed that higher order RK methods are more accurate - especially for large-size integration steps - than the others are. Author shows that the 5th order method or modified RKF methods are more accurate than the RK4 recommended by the GLONASS-ICD. On the other hand due to the simplicity of equations RK4 order method is the fastest of the all analysed methods. However, an argument of economical saving time was more important in the 90s, when PCs' computing power was much smaller than today. Currently due the highest accuracy of analysed methods more suitable for calculation of GLONASS satellite position is Runge-Kutta-Fehlberg 5th order method.

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