# Some Fundamental Properties of Transmission Systems*

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***Summary***—**The problem of the minimum loss in relation to the singing point is investigated for generalized transmission systems that must be stable for any combination of passive terminating impedances. It is concluded that the loss may approach zero db only in those cases where the image impedances seen at the ends of the system are purely resistive. Moreover, in such cases, the method of overcoming the transmission loss, whether by conventional repeaters or by series and shunt negative impedance loading, or otherwise, is quite immaterial to the external behavior of the system as long as the image impedances are not changed. The use of impedance-correcting networks provides one means of insuring that phase of the image impedance of the over-all system approaches zero.**

**General relations are derived which connect the image impedance and the image gain of an active system with its over-all performance properties.**

SINCE THE TIME when amplifiers first were introduced into the telephone plant, the properties of two-way repeaters have been subjected to extensive analysis. From this it might be inferred that further study is likely to uncover very little that is not already known. Nevertheless, it frequently happens that new types and permutations of repeater and loading circuits are proposed, and current methods of analysis are found to be quite difficult.

In the face of this situation, the present paper is intended to review the underlying fundamentals, and to present them in what is hoped to be a form that will allow them to be simply and easily applied in determining the over-all performance. In a wider sense, what is attempted is to state certain basic physical properties and limitations in a way that allows one to say, "Regardless of detail, if these rules are violated, it follows that the circuit cannot perform as predicted," or, on the other hand, to say, "The ideal performance of such-and-such a system is so-and-so. If the proposed plan does not approximate this ideal, it must be possible to find a better one."

As sometimes happens, this review of the properties of transmission systems has led to several concepts which are thought to be new. Their importance becomes more pronounced in connection with the current tendency to reduce the net operating loss of telephone systems to lower values than were customary in the past.

In the case of the telephone repeater, the extent of the various combinations and permutations that are encountered in practice has made difficult the statement of generalizations in simple terms. The present attempt is based on the development of linear network theory in respect to active four-poles that has been progressing perhaps quietly but nonetheless steadily in the past years. Like most mathematical generalizations, the solution of one problem is really the solution of a class of problems, and it will be found that in their broadest form, the generalizations which are now presented are just as applicable to the case of four-wire telephone and radio systems as they are to the conventional two-way repeater.

The system to be considered may contain repeaters of the 22-type, such as is illustrated in Fig. 1, or it may contain any of the other varieties. Moreover, there is no restriction placed on whether the gain is the same in both directions or not, and sections of line or of other

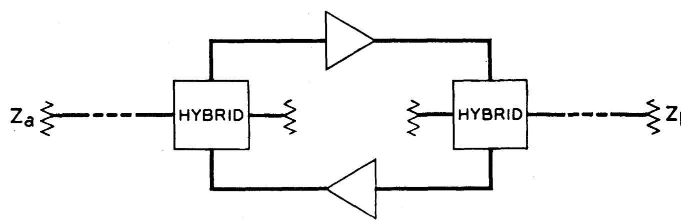


Fig. 1—Schematic of 22-type repeater.

circuit networks may be included as part of the unit under consideration. Even more broadly, the unit considered may consist either of a single repeater section, or of an unlimited number of repeater sections in tandem comprising an entire system. Restrictions are placed on these broad limits only in dealing with specific applications.

The analysis then directs itself to the general linear four pole such as is illustrated in Fig. 2 where the rectangular box may contain as much or as little as meets the needs of the particular situation. When terminations are added, the diagram illustrates the situation.

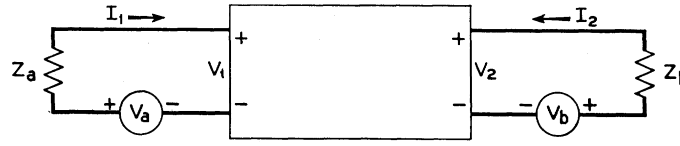


Fig. 2—Diagram of linear four-pole with terminations.

The equations describing Fig. 2 may be written:

$$
\begin{aligned}
Z_{11}I_1 + Z_{12}I_2 &= V_1 \\
Z_{21}I_1 + Z_{22}I_2 &= V_2
\end{aligned} \tag{1}
$$

where the $Z$'s are characteristic of the four-pole only, and do not involve the terminations. More will be said later about their properties and how they are derived.

* Decimal classification: 621.385. Original manuscript received by the Institute, May 12, 1951; revised manuscript received, October 15, 1951.

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The corresponding equations including the terminations may be written down immediately by noting that the terminations $Z_a$ and $Z_b$ are related to the currents and voltages by the formulas

$$\begin{aligned} V_1 &= V_a - Z_a I_1 \\ V_2 &= V_b - Z_b I_2. \end{aligned} \tag{2}$$

When combined with (1) these give

$$\begin{aligned} (Z_{11} + Z_a)I_1 + Z_{12}I_2 &= V_a \\ Z_{21}I_1 + (Z_{22} + Z_b)I_2 &= V_b, \end{aligned} \tag{3}$$

which may be solved for the currents,

$$I_1 = \frac{V_a(Z_{22} + Z_b) - V_b Z_{12}}{\Delta} \tag{4}$$

$$I_2 = \frac{(Z_a + Z_{11})V_b - Z_{21}V_a}{\Delta} \tag{5}$$

where

$$\Delta = Z_{11}Z_{22} - Z_{12}Z_{21} + Z_a Z_{22} + Z_b Z_{11} + Z_a Z_b \tag{6}$$

is the determinant of the system of (3). It will be noted, and later use will be made of the fact, that the determinant of (1) does not depend on the terminations, and is given by

$$\Delta_0 = Z_{11}Z_{22} - Z_{12}Z_{21}, \tag{7}$$

and, consequently, that (6) may be written

$$\Delta = \Delta_0 + Z_a Z_{22} + Z_b Z_{11} + Z_a Z_b. \tag{8}$$

When the four-pole of Fig. 2 is driven from the left, $V_b$ may be set equal to zero in (4) and (5). Under these conditions, the generator $V_a$ sees the internal impedance $Z_a$ in series with the impedance presented by the four-pole. From (4) we have then

$$V_a = I_1 \frac{\Delta}{Z_{22} + Z_b} \cdot \tag{9}$$

But we can write

$$V_a = I_1(Z_a + Z_A) \tag{10}$$

where $Z_A$ is the input impedance of the four-pole when it is terminated by $Z_b$. It results from (9) and (10) that

$$Z_A = \frac{\Delta}{Z_{22} + Z_b} - Z_a. \tag{11}$$

or, by (8),

$$Z_A = \frac{\Delta_0 + Z_b Z_{11}}{Z_{22} + Z_b} \cdot \tag{12}$$

An exactly similar procedure based on driving the four-pole from the right instead of from the left, gives the impedance seen looking into that end when the left-hand termination is $Z_a$. The result is:

$$Z_B = \frac{\Delta_0 + Z_a Z_{22}}{Z_{11} + Z_a} \cdot \tag{13}$$

From these last two relations, it is possible to find the impedance values for the terminations $Z_a$ and $Z_b$ that would simultaneously match the impedances $Z_A$ and $Z_B$. These are the so-called image impedances, and are found by putting

$$\begin{aligned} Z_a &= Z_A = Z_{\mathrm{I}} \\ Z_b &= Z_B = Z_{\mathrm{II}}, \end{aligned}$$

and solving (12) and (13) simultaneously. The result is:

$$Z_{\mathrm{I}} = \sqrt{\frac{Z_{11}}{Z_{22}}\Delta_0} \tag{14}$$

$$Z_{\mathrm{II}} = \sqrt{\frac{Z_{22}}{Z_{11}}\Delta_0}. \tag{15}$$

When the terminations have these values, there are no reflections from the terminating impedances (although there may be internal reflections within the four-pole) and, in the cases where the image impedances (14) and (15) are pure resistances, the gain in power resulting from the presence of the four-pole is a maximum.

Concerning these power relationships, there is a good deal more that needs to be said. In the first place, it turns out to be more convenient to deal in terms of "virtual" power rather than real power. The difference is that the former is given by $I^2Z$ in general, even when the currents are represented by complex numbers involving imaginaries, while the latter is equal to the product of the square of the magnitude of the current times the resistive component of the impedance. Consequently, the writing is greatly simplified by the concept of virtual power, whereas the real power may be found from it when that is required, and the phase of the terminations is known.

When the four-pole is driven from the left, so that $V_b$ may be put equal to zero in (5), the virtual power in the output termination $Z_b$ is given by

$$I_2^2 Z_b = V_a^2 \frac{Z_{21}^2}{\Delta^2} Z_b = V_a^2 \frac{Z_{21}}{Z_{12}} \frac{Z_{12}Z_{21}}{\Delta^2} Z_b. \tag{16}$$

The operating gain is defined as the ratio of this to the virtual power that the generator $V_a$ would deliver directly to a matched load, $Z_a$. Thus, when the generator $V_a$ is connected to a matched load, the current is $V_a/2Z_a$, and the virtual power in the load is $V_a^2/4Z_a$. The operating gain[1] is therefore

$$\Gamma_{21} = \frac{Z_{21}}{Z_{12}} \cdot \frac{Z_{12}Z_{21}}{\Delta^2} \cdot 4Z_a Z_b \tag{17}$$

where symbol $\Gamma_{21}$ indicates that the gain is from left to right. In the opposite case, where the four-pole is driven from $V_b$ on the right while the virtual output power is absorbed in $Z_a$ on the left, the corresponding expression for operating gain is

[1] The insertion gain may be found by multiplying the operating gain by

$$\frac{(Z_a + Z_b)^2}{4Z_a Z_b}.$$

$$\Gamma_{12} = \frac{Z_{12}}{Z_{21}} \frac{Z_{12}Z_{21}}{\Delta^2} 4Z_aZ_b. \tag{18}$$

It is obvious, therefore, that the ratio of the gains in the two directions is $(Z_{21}/Z_{12})^2$. When $Z_{12}$ is equal to $Z_{21}$, the gain (or loss, which is the reciprocal of the gain) in the two directions is likewise the same.

The expressions (17) and (18) are not particularly complicated, but for physical interpretation they may be put into very much better shape. This requires a little algebra, but, to make our proofs complete, it is worth outlining the procedure in some detail, rather than merely stating the final result.

The first two steps may probably be combined into one without impossing undue difficulties. Thus, from (7) the expression $Z_{12}/Z_{21}$ may be replaced by $Z_{11}Z_{22}-\Delta_0$. This is the first step. The next one is a matter of definition, and merely eliminates $Z_a$ and $Z_b$ by introducing the ratios

$$\begin{aligned} a &= Z_a/Z_{\mathrm{I}} \\ b &= Z_b/Z_{\mathrm{II}}. \end{aligned} \tag{19}$$

When these substitutions are made in (17), remembering that $\Delta$ is given by (8), we have, with the help of (14) and (15),

$$\begin{aligned} \Gamma_{21} &= \frac{Z_{21}}{Z_{12}} \frac{[Z_{11}Z_{22}-\Delta_0]4ab\Delta_0}{[\Delta_0(1+ab)+(a+b)\sqrt{Z_{11}Z_{22}\Delta_0}]^2} \\ &= \frac{Z_{21}}{Z_{12}} \frac{\left[1-\dfrac{\Delta_0}{Z_{11}Z_{22}}\right]4ab}{\left[a+b+(1+ab)\sqrt{\dfrac{\Delta_0}{Z_{11}Z_{22}}}\right]^2} \\ &= \frac{Z_{21}}{Z_{12}} \frac{1-\sqrt{\dfrac{\Delta_0}{Z_{11}Z_{22}}}}{1+\sqrt{\dfrac{\Delta_0}{Z_{11}Z_{22}}}} \left[\frac{1+\sqrt{\dfrac{\Delta_0}{Z_{11}Z_{22}}}}{a+b+(1+ab)\sqrt{\dfrac{\Delta_0}{Z_{11}Z_{22}}}}\right]^2 4ab. \end{aligned} \tag{20}$$

In the event that the terminations on input and output sides are matched to the image impedances, so that $a$ and $b$ are both equal to unity, the gain from (20) is given by

$$\Gamma_{21}' = \frac{Z_{21}}{Z_{12}} \frac{1-\sqrt{\dfrac{\Delta_0}{Z_{11}Z_{22}}}}{1+\sqrt{\dfrac{\Delta_0}{Z_{11}Z_{22}}}}, \tag{21}$$

which is often written in the alternative form

$$\Gamma_{21}' = \frac{Z_{21}}{Z_{12}} \frac{1-\tanh\theta}{1+\tanh\theta}$$

where $\theta$ is the propagation constant of the four-pole. It is convenient to write this matched gain in the more condensed form

$$\Gamma_{21}' = \frac{Z_{21}}{Z_{12}} \Gamma_0$$

where the image gain $\Gamma_0$ is defined by the relation

$$\Gamma_0 = \frac{1-\sqrt{\dfrac{\Delta_0}{Z_{11}Z_{22}}}}{1+\sqrt{\dfrac{\Delta_0}{Z_{11}Z_{22}}}}. \tag{22}$$

The quantities under the radical may then be written as follows by solving (22):

$$\sqrt{\frac{\Delta_0}{Z_{11}Z_{22}}} = \frac{1-\Gamma_0}{1+\Gamma_0}. \tag{23}$$

For its physical meaning, note that in the matched condition, $\Gamma_0$ is the geometric mean between the gains in the two directions.

Substitution of (23) into (20) gives:

$$\Gamma_{21} = \frac{Z_{21}}{Z_{12}} \Gamma_0 \frac{4a}{(1+a)^2} \cdot \frac{4b}{(1+b)^2} \cdot \frac{1}{\left(1-\Gamma_0 \dfrac{1-a}{1+a} \cdot \dfrac{1-b}{1+b}\right)^2}. \tag{24}$$

The expression (24) is now in the form which we were seeking. Its advantage is the physical interpretation which may be given to factors of the form

$$\frac{4a}{(1+a)^2} \quad \text{and} \quad \frac{1-a}{1+a}.$$

The first of these might be called the "mismatch" factor, and expresses the ratio of the virtual power which a generator puts into a load connected directly across its terminals to the virtual power it would put into a matched load similarly connected. The situation is well known for the case where $a$ is a real number, and calculation illustrates how slowly the gain departs from its matched value as the impedance ratio departs from unity. For example, a two-to-one impedance mismatch means a loss of 0.5 db only. Even a ten-to-one mismatch gives only 4.8 db loss. Note, too, the curve is symmetrical about the value of unity for the impedance ratio.

The other factor is the ratio of the reflected to the incident current at the end of a line terminated by an impedance mismatch. Its reciprocal is thought to constitute a more precise definition of "return loss" than is usually given in current literature. Note also that the two factors are related through the equation

$$\frac{4a}{(1+a)^2} + \left(\frac{1-a}{1+a}\right)^2 = 1, \tag{25}$$

which states the physical fact that the sum of the absorbed power and the reflected power is equal to the incident power.

With these relations in mind, it is possible now to interpret the various factors in (24) in connection with the diagram of Fig. 2. Imagine the generator $V_a$ to send a wave into the four-pole represented by the rectangle in the drawing. Disregarding the factor $Z_{21}/Z_{12}$ for the moment, we can visualize the wave as progressing from the generator $V_a$ toward the right until it meets the impedance discontinuity between $Z_a$ and $Z_I$, the image impedance of the four-pole seen from the left. Of the virtual power in the incident wave, the fraction $4a/(1+a)^2$ progresses on into the four-pole while the remainder is reflected and lost in the generator impedance. Having entered the four-pole, the current wave is amplified by the factor $\sqrt{\Gamma_0 Z_{21}/Z_{12}}$, and emerges from the right-hand end of the rectangle. Here another impedance discontinuity is encountered and the fraction $4b/(1+b)^2$ of the power enters the load, while the fraction $(1-b)/(1+b)$ of the current is reflected and progresses back toward the left through the four-pole. The current is amplified by the amount $\sqrt{\Gamma_0 Z_{21}/Z_{12}}$, is reflected in part by the factor $(1-a)/(1+a)$ at the left-hand termination, and moves once more toward the right. Thus, within the four-pole there is set up a to-and-fro surging which, each time the wave arrives at the right, contributes a little more to the power in the output.

In a single round trip through the four-pole, the wave of current or voltage is modified by the factor

$$\sqrt{\Gamma_0 \frac{Z_{21}}{Z_{12}}}\sqrt{\frac{Z_{12}}{Z_{21}}\Gamma_0}\frac{1-a}{1+a}\cdot\frac{1-b}{1+b} = \Gamma_0\frac{1-a}{1+a}\cdot\frac{1-b}{1+b},$$

and the sum of an infinite number of round trips assumes the form

$$S = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \qquad (26)$$

where

$$x = \Gamma_0\frac{1-a}{1+a}\cdot\frac{1-b}{1+b},$$

and when $|x|<1$. The square of the sum must be taken in (24) because $S$ represents a current, while (24) represents a power ratio. It is thus seen then that all of the factors in (24) may be accounted for on a physical basis, and the whole action may consequently be thought of in pictorial perspective. The usefulness of introducing the image gain $\Gamma_0$, which is the geometrical mean of the forward and reverse gains, has also been demonstrated in this connection.

However, its usefulness does not stop with (24), and the impedances presented by the four-pole may also be expressed in terms of $\Gamma_0$. Thus (12) and (13) may be written respectively, with the help of (23):

$$\frac{Z_A}{Z_I} = \frac{1-\Gamma_0\dfrac{1-b}{1+b}}{1+\Gamma_0\dfrac{1-b}{1+b}} \qquad (27)$$

$$\frac{Z_B}{Z_{II}} = \frac{1-\Gamma_0\dfrac{1-a}{1+a}}{1+\Gamma_0\dfrac{1-a}{1+a}}\cdot \qquad (28)$$

These show immediately that the impedance presented by the four-pole becomes the same as the image impedance whenever $a=1$ in the one case, or $b=1$ in the other. This is, of course, axiomatic. A much more striking property is shown by noting that the intrinsic algebraic sign of the impedance must perforce be the same as that of the image impedance whenever the magnitude of $\Gamma_0(1-b)/(1+b)$ in the one case, and of $\Gamma_0(1-a)/(1+a)$ in the other, is less than unity. The converse is true when the magnitudes are greater than unity, so that whenever the image gain is sufficiently large, the input impedance is the negative of the image impedance unless $a$ or $b$, as the case may be, is identically unity.

When $\Gamma_0=1$, it is interesting to note that $Z_A=Z_bZ_I/Z_{II}$, and when $\Gamma_0=-1$, that $Z_A=Z_IZ_{II}/Z_b$.

Having dealt now with the derivation and discussion of expressions for impedances and gains, we come to the very important question of stability, that is, freedom from oscillation. This may be approached in several ways, but the most rigorous is probably to return to the general equations (3), and their solutions given by (4) and (5). From (4) and (5) it is seen that the currents $I_1$ and $I_2$ may be different from zero even in the absence of the driving sources $V_a$ and $V_b$ whenever $\Delta=0$. But $\Delta$ is a function of all of the internal network impedances as well as of the terminations $Z_a$ and $Z_b$. In turn, all of these impedances are functions of $j\omega$. For purposes of analysis, $j\omega$ may be replaced by the more general variable $p=\alpha+j\omega$, so that currents and voltages of the form $e^{j\omega t}$ now become $e^{pt}=e^{(\alpha+j\omega)t}$. The significance of $\alpha$ then becomes apparent. When it is positive, the currents and voltages increase indefinitely with time. When it is zero, they are the usual sinusoids of constant amplitude, and when it is negative, the currents and voltages decrease with time and eventually die away altogether.

For stability it is evident that the relation $\Delta=0$ must be satisfied for negative values of $\alpha$ only, and not for positive values, as otherwise the currents in (4) and (5) would increase indefinitely with time, even in the absence of the driving sources $V_a$ and $V_b$. If the equation $\Delta=0$ is satisfied only for negative values of $\alpha$, the currents die away when the sources are removed and the system is stable, except in the contingency that one of the coefficients of $V_a$ or $V_b$ in (4) or (5) should become infinite for some positive value of $\alpha$ while, at the same time, $\Delta$ itself remained finite. Since $\Delta$ may be written

$$\Delta = (Z_{11}+Z_a)(Z_{22}+Z_b) - Z_{12}Z_{21},$$

and consequently involves all of the aforementioned coefficients, $\Delta$ can remain finite when one of the coefficients becomes infinite only if the coefficient with which it is paired in the above expression for $\Delta$ becomes zero simultaneously or (a more usual situation) is identically zero for all values of $p$. That is, either $(Z_{11}+Z_a)$ is infinite for the same value of $p$ that causes $(Z_{22}+Z_b)$ to become zero, or vice versa, or else $Z_{12}$ is infinite for the same value of $p$ that causes $Z_{21}$ to become zero. In either event, instability would require that the real part of $p$ should be positive. This alternative contingency seldom occurs in bilateral systems, but is not infrequently encountered in unilateral cases. One particular example that is illustrative happens when the interstage coupling circuit between two unilateral amplifier stages contains negative impedances and, when isolated, is unstable. Connecting it between two vacuum tubes does not cause it to become stable, and it will be found that the four-pole equations for the system show that $Z_{12}$ is zero for all frequencies, but that $Z_{21}$ may become infinite for a positive value of $\alpha$.

Whenever (3) is derived by first writing the mesh equations for the entire multi-mesh network, one equation for each mesh, and from these equations eliminating all currents but the two corresponding to the input and output meshes, the stability conditions are completely determined either by the vanishing of $\Delta$, or by the simultaneous vanishing of one of a pair of factors forming $\Delta$ together with the vanishing of the reciprocal of the other.

Possibility of trouble occurs, however, when approximations are made. For example, when a vacuum tube with feedback is considered, the impedance looking into a pair of terminals may become negative in certain frequency ranges. There is then a strong inclination to simplify by replacing the complete details of the circuit which produced the negative impedance by the negative impedance itself. Actually, there is no objection to doing this providing that the negative impedance is completely and accurately specified over the whole frequency range.

This point is very important. For example, note that a negative impedance which was the exact negative of some passive impedance over the whole frequency range from zero to infinity, could not possibly be unstable on either open or short circuit. This is at once evident when it is considered that the values of $p$ which satisfy the passive equation

$$Z(p) = 0$$

are identical with those that satisfy the active equation

$$-Z(p) = 0$$

and hence, if the $\alpha$ for the one is always negative, so also is the $\alpha$ for the other. From this it may further be concluded that a negative impedance which is unstable on either open or short circuit cannot possibly be the exact negative of any passive impedance whatever over the whole frequency range. One can go even further, however, and invoke some of the methods of complex function theory to show that such a negative impedance cannot even be the exact negative of any passive impedance over any finite frequency band, no matter how small.

The point of this discussion is to bring out the fact that stability or lack of it in systems involving negative impedances is often determined by the departure of the negative impedances from being the negatives of passive impedances, and hence that any disregard of this fundamental fact is likely to lead to trouble. These departures may, and in fact often do, exist at frequencies outside of the band that is of interest from the standpoint of normal use. Their effect reflects back into that band nonetheless.

How then should one proceed? Is the device of using the concept of negative impedances of no practical value? The answer to this is supplied in part by Crisson, who, some years ago, introduced the concept of series and shunt types of negative resistances. By definition, the series type is unstable on short circuit, and the shunt type is unstable on open circuit. Interpreted in the light of the foregoing discussion, these definitions may be rephrased somewhat as follows:

A negative resistance is one which behaves very nearly like the negative of a positive resistance over a fairly large frequency range. Outside of that range, however, a series type negative resistance departs from that approximation in such a way that the circuit element is unstable on short circuit, and a shunt type negative resistance departs in such a way that the circuit element is unstable on open circuit. Graphically, this would imply that, if the imaginary part of the negative impedance were plotted against the real part for all values of frequency, that is for all values of $p = \alpha + j\omega$ where $\alpha = 0$, the graph of a series type would look something like Fig. 3, and the graph of a shunt type would look something like Fig. 4. They both encircle the origin, but in

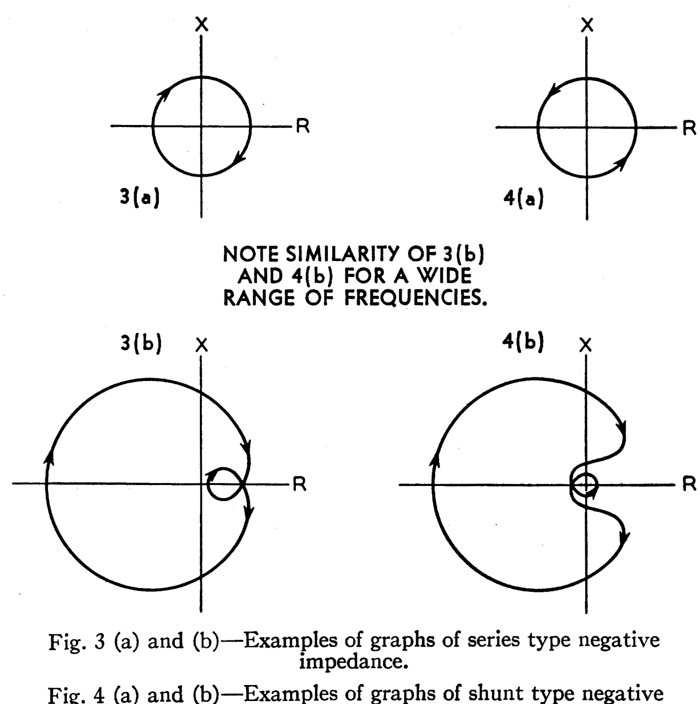


Fig. 3 (a) and (b)—Examples of graphs of series type negative impedance.

Fig. 4 (a) and (b)—Examples of graphs of shunt type negative impedance.

different directions. It is obvious that this approximation, useful as it is, has certain limitations, and that the safest way of dealing with new or untried circuits is to be sure that the negative impedance is, in fact, specified to a sufficient extent over the whole pertinent frequency range. Such a range would have to be sufficient to insure that the combination of the negative element with the remainder of the circuit did not have a resistive component that became negative at any higher or any lower frequency.

The practical effect of all of this is to point out that certain combinations of series and shunt type negative resistances may be quite stable, while others may not. The general stability criterion, when all things are taken into account, is the determination of the values of $p$ that cause $\Delta$ in (4) or (5) to become zero. The alternative condition that results in instability can usually be detected by general inspection of the circuit, or may be tested for each of the four possible contingencies separately.

The investigation of $\Delta$ itself turns out to be rather cumbersome, and an easier alternative arises when it is noticed that the expression for gain $\Gamma$, given by (17), contains $\Delta$ in the denominator. It follows that $\Gamma$ has an infinity whenever $\Delta$ has a zero. Also, $\Gamma$ has no infinities that are not contributed by zeros of $\Delta$. This may be verified by inserting (6) into (17) and noting that infinities of $Z_a$ and $Z_b$ contribute only zeros to $\Gamma$, while infinities of $Z_{12}$ and $Z_{21}$ contribute neither zeros nor infinities to $\Gamma$. Consequently, except for the case mentioned before where $Z_{21}/Z_{12}$ has an infinity while $Z_{12}Z_{21}$ does not the zeros of $\Delta$ are uniquely determined by the infinities of $\Gamma$ and, when $\Delta$ has no zeros with positive real parts, $\Gamma$ has no infinities with positive real parts. For every zero of $\Delta$ that does have a positive real part, $\Gamma$ has an infinity with a positive real part.

It may be taken then that, leaving aside the exceptions mentioned, (17) can be used as a basis for determining stability, and therefore that (24), which is merely (17) written in another form, can likewise be used. The infinities of (24) must be investigated to determine whether the real parts of any of them are positive.

The possible infinities of $\Gamma$ are all determined by the equation

$$\left(1 - \Gamma_0 \frac{1-a}{1+a}\cdot\frac{1-b}{1+b}\right) = 0, \tag{29}$$

as may be seen from (24) by trying all of the other alternatives; namely $1/a=0$, $1/b=0$, $\Gamma_0=0$; $(1+a)=0$, and $(1+b)=0$. None of these others yields infinities.

There is a striking similarity between the form of (29) and the famous equation for the stability of feedback amplifiers, usually written

$$(1 - \mu\beta) = 0.$$

In fact, the similarity goes further than one of form only, and the discussion leading to (26) shows that the physical meaning of the factors involved is quite analogous. This at once suggests the possibility of applying the Nyquist stability criterion and plotting

$$\Gamma_0 \frac{1-a}{1+a}\cdot\frac{1-b}{1+b}$$

on the complex plane as a function of the frequency $\omega$, and seeing whether the plot encircles the point $(1, j0)$. The trouble is that encirclement of the point $(1, j0)$ would indicate instability only under certain special conditions, and cannot be applied with complete generality. It happens that those conditions are fulfilled in the standard type of feedback amplifier, but very often are not in the more general cases which it is now attempted to discuss.

This fact is so important, and the appreciation of it seems so limited in extent, that a brief explanation of the fundamentals involved appears to be in order. The key to the situation is furnished by the realization that, in the conventional feedback amplifier, both $\mu$ and $\beta$ are of the nature of constants multiplied by the ratio of output to input voltage across passive impedance functions (either self or transfer) and hence that neither of them has infinities whose real parts are positive. In the generalized repeater case of (29), where negative impedance elements may be involved, there is no assurance that this is so. In fact, it is readily seen that the reflection coefficient $(1-a)/(1+a)$ may become infinite when a negative resistance is connected facing a positive one, for then $a$ is negative. This seems at first to be very discouraging to an attempt to draw simple conclusions and rules relating to the more general case. The situation is helped only by limiting the problem and being content, not with complete generality, but with an amount sufficient to cover the particular class of problem that is encountered in considering the telephone repeater. Here this analysis requires broadly, not only that the system be stable with a given pair of terminations, but that it be stable when its end terminations are either open circuited or short circuited in any possible combination of the terminations, and, moreover, that it shall be stable for any values of passive terminations in between these two extremes.

Further, it is evident in such a system that the ultimate terminations at the final terminals must consist of passive impedances. This at once implies that the image impedances of the four-pole representing the entire system must likewise have the properties of a passive impedance, as otherwise it may be shown, from (12) for example, that there always exists a value of passive termination that will result in instability. This is really a very important conclusion for it says that, in the design of systems involving negative impedances, care must be taken that the image impedances must have these passive properties at all frequencies if stability is to be guaranteed. This means that their resistive components must be positive at all frequencies from zero to infinity.

If the image impedances were entirely resistive at all frequencies, while the terminations were restricted to being passive, the greatest as well as the least magnitude that could be attained by factors of the form $(1-a)/(1+a)$ would occur when the termination approached a pure reactance, either positive or negative. The magnitude of the factor would then be unity for any value of terminating reactance. Its phase, however, could lie in any of the four quadrants of the complex impedance plane, depending upon the value of terminating reactance. Hence, when the magnitude of the image gain passed through unity, and in the event that the gain factor $\Gamma_0$ had even the smallest phase angle, it always would be possible to find values of terminating reactance that would cause the graph of

$$\Gamma_0 \frac{1-a}{1+a} \cdot \frac{1-b}{1+b}$$

to pass through the point $(1, j0)$ on the complex impedance plane. Minute changes in terminating reactance would then cause the graph to pass on one side or the other of the point $(1, j0)$ and consequently change the system from a stable one to an unstable one, or vice versa.

For this case, where the image impedances are pure resistances at all frequencies, it is clear that the system will be stable for any values of passive terminating impedances if and only if the magnitude of the image gain is less than unity. This is the situation that can be approached by the 22-type repeater of Fig. 1 when its image impedances are pure resistances, though even here departures of the hybrid coils from the ideal can introduce phase into the image impedances and create the more general situation which must now be discussed.

In this more general case, the restriction that the resistive components of the image impedances must be positive at all frequencies is still retained, as otherwise passive terminations which will cause singing can always be found. However, no restriction is now placed on the reactive component of the image impedanc . Under these conditions, it can be shown from a theorem in functions of complex variables[2] that factors of the form $(1-a)/(1+a)$ attain their greatest and their least magnitudes as well as their greatest and smallest real and imaginary components when the terminations are pure reactances. For this condition, we can write:

$$\frac{1-a}{1+a} = \frac{Z_I - Z_a}{Z_I + Z_a} = \frac{R_I + j(X_I - X_a)}{R_I + j(X_I + X_a)}$$

$$= \sqrt{\frac{1 + y^2 - 2y \sin \phi_I}{1 + y^2 + 2y \sin \phi_I}}\, e^{-j\tan^{-1}(2y\cos\phi_I/(1-y^2))}, \quad (30)$$

where

$$y = X_a/|Z_I| \quad \text{and} \quad \phi_I = \tan^{-1} \frac{X_I}{R_I} \cdot$$

This attains its greatest magnitude when $|X_a| = |Z_I|$. and the algebraic sign of $X_a$ is opposite to that of $X_I$, In that event, the magnitude becomes

$$\left|\frac{1-a}{1+a}\right|_{\max} = \sqrt{\frac{1 + |\sin \phi_I|}{1 - |\sin \phi_I|}}, \quad (31)$$

and the phase is $\pm\pi/2$ depending upon the phase of $Z_I$.

[2] H. W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand, 1945; p. 169.

The minimum magnitude is the reciprocal of (31), or

$$\left|\frac{1-a}{1+a}\right|_{\min} = \sqrt{\frac{1 - |\sin \phi_I|}{1 + |\sin \phi_I|}}, \quad (32)$$

and occurs at an angle of $\pi$ with respect to that for the maximum. The real component of (30) attains its maximum value when

$$y = -\frac{1 \mp \cos \phi_I}{\sin \phi_I} \cdot \quad (33)$$

A graph of (30) for the four cases where the phase of $Z_I$ is 0, 30°, 45° and 60°, respectively, is shown on Fig. 5 which illustrates the locus of the function as $X_a$ takes on all values from $-\infty$ to $+\infty$. Further study of this figure, and the equations above relating to it, shows that the curves are true circles and that the distance from the origin to the center of a given circle is equal to $\tan \phi$, where $\phi$ is the phase of the corresponding image impedance.

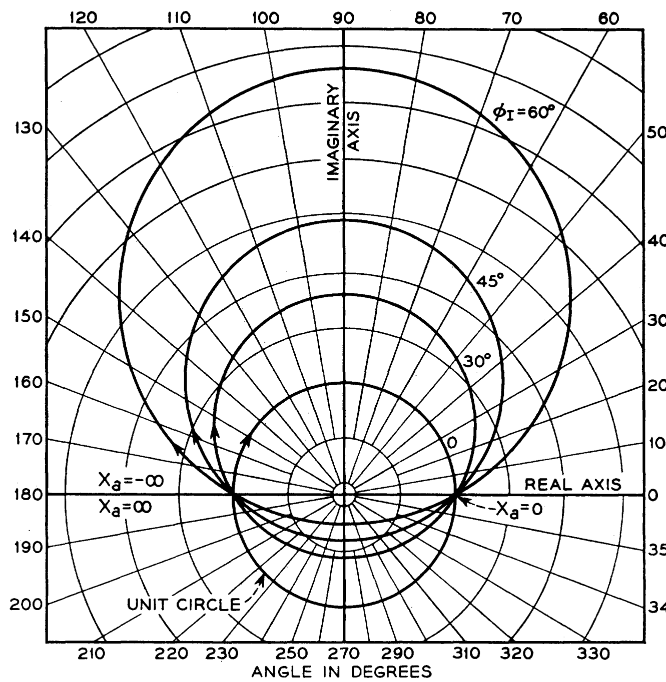


Fig. 5—Locus of $(1-a)/(1+a)$ for increasing values of $X_a$, where $a = jX_a/Z_I$.

At the same frequency as that for which the graph of $(1-a)/(1+a)$ has been drawn in Fig. 5, the graph of $(1-b)/(1+b)$ may be constructed from the properties of $Z_{II}$, the image impedance at the output terminals of the network. Where the two image impedances $Z_I$ and $Z_{II}$ are the same, the graph of $(1-b)/(1+b)$ is a duplicate of that of $(1-a)/(1+a)$. For any combination of terminating reactance values $X_a$ and $X_b$, the product of the two graphs always falls within the envelope obtained by letting $X_a = X_b$. For this condition, Fig. 6 shows the product curve for several values of the phase of the image impedance, and it will be noted that the external envelope of the complete surface is always equal to or greater than unity.

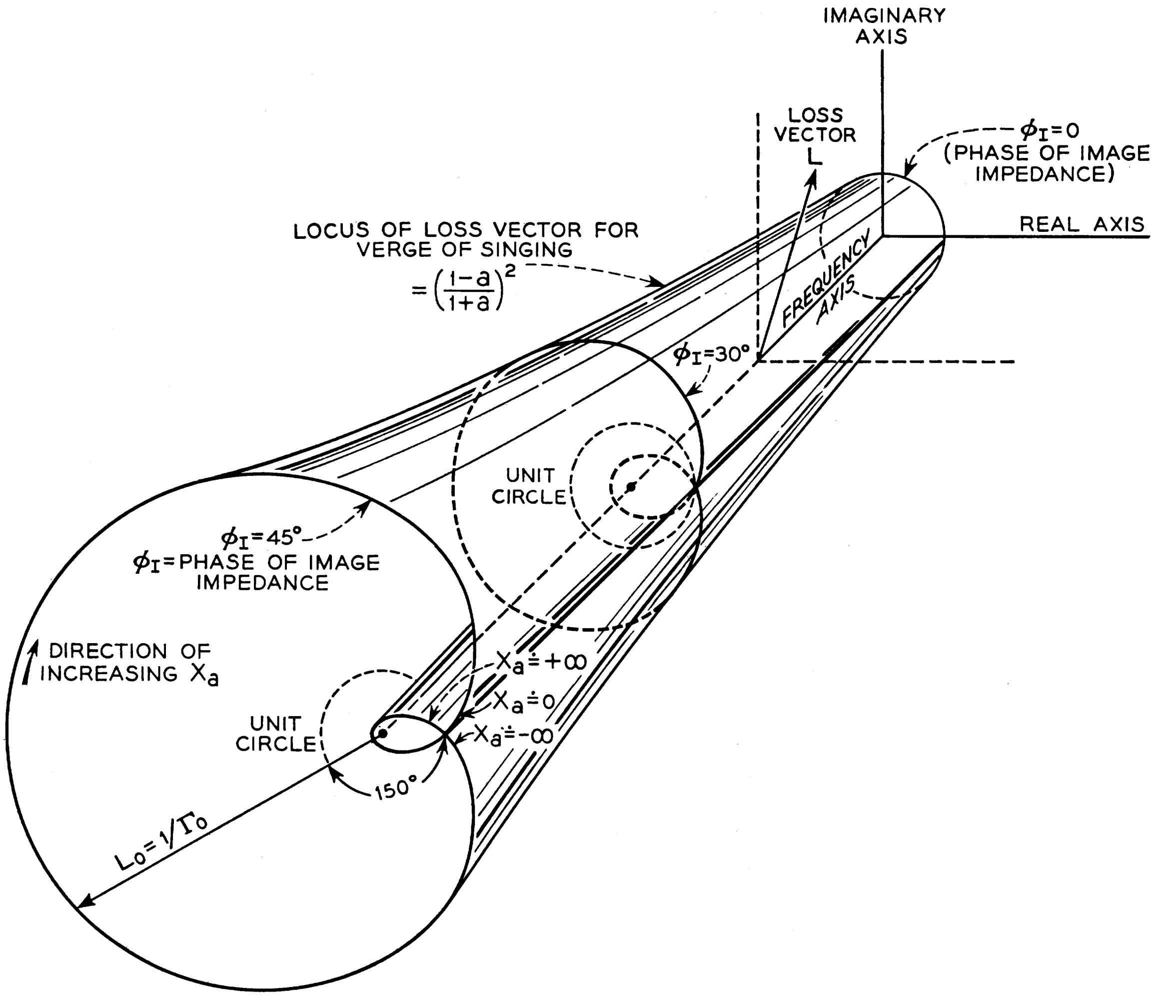


Fig. 6—Locus of stability for the loss vector for different frequencies.

Suppose, at the frequency for which the largest curve in Fig. 6 is drawn, that the gain $\Gamma_0$ has, for example, a phase of 150°. The image loss, which is the reciprocal of $\Gamma_0$, then has a phase of $-150°$. If it had a magnitude equal to that of the vector shown on the figure, the product

$$\Gamma_1 \frac{1-a}{1+a} \cdot \frac{1-a}{1+a}$$

would be exactly equal to unity, and hence, according to (29), the system would be on the verge of singing. A smaller gain at the same phase would be needed for stability or else, for this case, the same gain at a lesser phase. Fig. 6 therefore sets the relation between the allowable phase and magnitude of the gain at the particular frequency it represents, and for the symmetrical case where the image impedances at both ends of the system are the same. A curve analogous to the one considered above must be constructed for every frequency, and the three-dimensional envelope of all of them determines the allowable relationship between the maximum magnitude and phase of the gain over the frequency range. Several such curves are shown on Fig. 6 for different values of the frequency and the phase of the image impedance. The only cases in which the magnitude of the gain can approach unity are first, those for which the image impedances are both pure resistances and, second, those for which the phase of the gain is exactly zero. In all other cases, the magnitude of the gain must be less than unity to avoid singing. When the phase of the gain is 180°, its magnitude must be less than

$$\frac{1-|\sin\phi_I|}{1+|\sin\phi_I|}.$$

When the image impedances are pure resistances, the gain can approach unity regardless of its phase. The three-dimensional surface shown in Fig. 6 can then be regarded as setting the lower limit on the loss. The end of the loss vector must always fall outside of this surface at every frequency.

In many practical cases, the phase of the gain is not under control. For example, the phase changes very rapidly with frequency in a circuit several hundred miles long, and it would not be feasible to attempt to keep it within narrow limits over the speech band. For these usual cases, the curve on Fig. 7, which is plotted from the above expression for the magnitude of the gain, gives the allowable operating condition. For example, when the phase angle of the image impedance is 60°, the loss must exceed 11.3 db. The required loss is, of course, in addition to the allowance of a margin to take care of such things as changes in amplifier gains and in line losses.

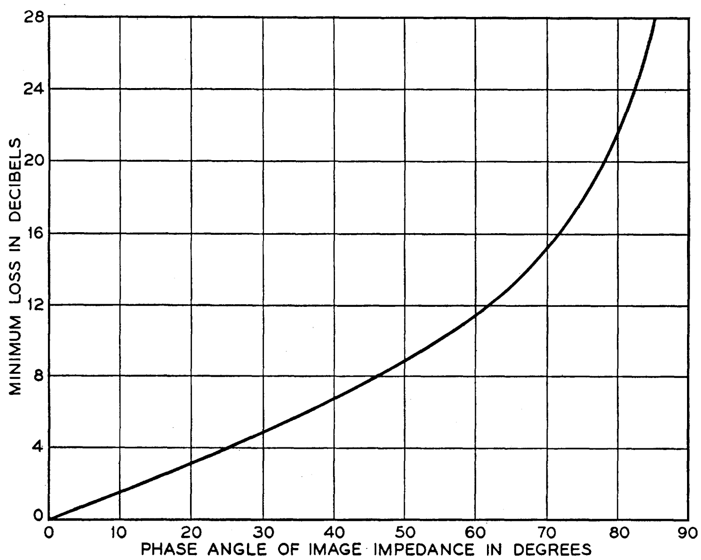


Fig. 7—Relation of minimum loss to phase angle of image impedance.

Of course, a magnitude of gain approaching unity is greatly to be desired in repeatered systems. In those which are composed of similar sections in tandem, and where, in addition, it is desired that the individual sections be stable when isolated and terminated with any combination of passive impedances, the above conditions become very important for they apply to the individual sections and severely restrict the freedom of design. Where, however, the sections need not be stable individually when subjected to all combinations of passive termination, but the system as a whole must nonetheless be stable, a good deal more freedom in design is permissible. The individual sections can actually have gains greater than unity, providing that it is removed before the final terminals are encountered.

For example, Fig. 8 shows a possible system in which repeaters or negative-resistance loading may be used quite freely in a transmission line, with the result that the system would sing if terminated at $Z_I$, with certain combinations of passive impedances. In general, also, the image impedances of the line will have a reactive component. At each end of the line there is placed an impedance-equalizing four-pole which matches the line on the one side, but presents a purely resistive impedance on the other. Such a network unavoidably introduces a certain loss if composed of passive elements only. An active network, however, such as the 22-type repeater circuit, can accomplish the impedance transformation without loss, or even with gain. In any event, the overall system, now having purely resistive image impedances, can be adjusted to have an overall gain that, with ideal impedance matches, approaches unity as closely as desired, and still will be completely stable for all combinations of passive termination. Any gain greater than unity, however, will result in singing, and the margin needed in practical design is a matter of how constant with time the values of the components of the system can be made, and how accurately the impedances may be matched.

In the event that the phase equalizers or converters of Fig. 8 are to be composed of passive circuit elements, the minimum loss required to consummate the impedance transformation can be found from Fig. 5. It is necessary to note that,

$$\Gamma_0 \frac{1-a}{1+a} \cdot \frac{1-b}{1+b}$$

can never pass through the point $(1, j0)$. For a reactive image impedance on the input side of the network, Fig. 5 would show the graph of the factor $(1-a)/(1+a)$ as one of the off-center circles. The graph of $(1-b)/(1+b)$ would be a unit circle, however, because of the resistive image impedance on the output side of the network, and the factor could have its phase anywhere in the four quadrants. The envelope of the product of the two factors would therefore be a circle whose radius vector was equal in magnitude to the maximum value of $(1-a)/(1+a)$, and whose phase could lie anywhere in the four quadrants. The minimum loss possible in a passive phase converter network would therefore be the

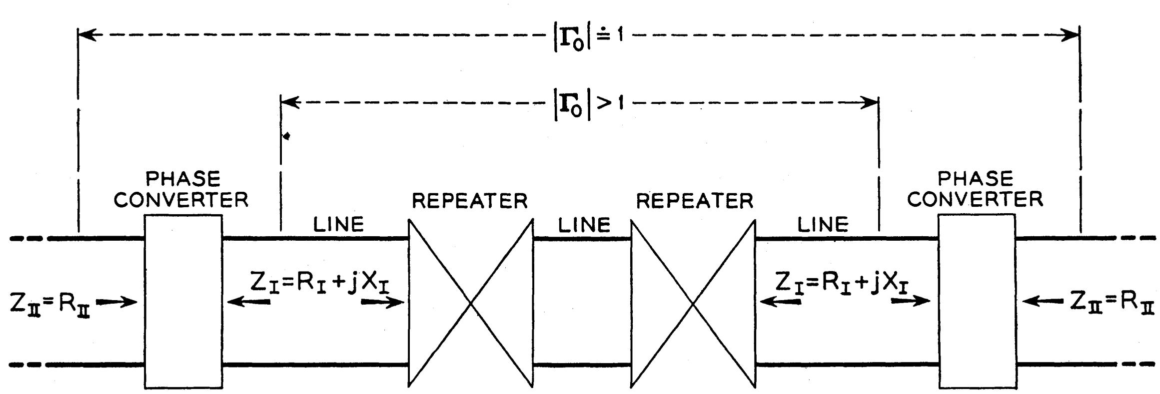


Fig. 8—Correction of phase of image impedance to increase over-all allowable gain.

reciprocal of this value, or

$$\sqrt{\frac{1+|\sin\phi_{\mathrm{I}}|}{1-|\sin\phi_{\mathrm{I}}|}}.$$

For systems in general, when the phase of the image impedances differ on the two ends, the stability conditions are in a sense more severe than for the symmetrical case. Here the criterion may be visualized by referring to the two curves on Fig. 5 corresponding to phases of the image impedances 30° and 60°. The first may be thought of as the reflection factor for the image impedance $Z_{\mathrm{I}}$, and the second as the reflection factor for the image impedance $Z_{\mathrm{II}}$. What is desired is an envelope analogous to Fig. 6, giving at each phase the maximum possible value of the product of the two factors. When a point on the envelope has a certain phase, $\psi$, the sum of the phases of the individual factors must be equal to $\psi$. Thus if $\rho_a$ represents the magnitude of $(1-a)/(1+a)$ and $\phi_a$ represents its phase, and if $\rho_b$ represents the magnitude of $(1-b)/(1+b)$ and $\phi_b$ represents its phase, the envelope at the angle $\psi$ has the magnitude $\rho_a\rho_b$ for which $\phi_a+\phi_b=\psi$. The problem is to determine the maximum magnitude of this product for each value of $\psi$.

The easiest approach seems to be to deal, not with $\rho_a$ and $\rho_b$ directly, but with their logarithms, so that we have

$$\log \rho_a\rho_b = \log \rho_a + \log \rho_b,$$

and the problem is shifted from that of finding the maximum value of a product to the somewhat easier one of finding the maximum value of a sum, subject, however, to the same condition concerning phase, namely,

$$\phi_a + \phi_b = \psi.$$

Fig. 9 is constructed from Fig. 5, and shows $\log \rho_a$ or $\log \rho_b$ plotted against $\phi_a$ or $\phi_b$, as the case may be, for different values of the phase $\phi_{\mathrm{I}}$ or $\phi_{\mathrm{II}}$ of the respective image impedance. The curves resemble sine waves somewhat, but are not true sinusoids although, for a rough approximation, the assumption that they are would not give large errors for moderately small value of $\phi_{\mathrm{I}}$.

$$\rho = \frac{1-a}{1+a}$$
$$a = jX_a/Z_{\mathrm{I}}$$
$$Z_{\mathrm{I}} = \text{image impedance}$$
$$= |Z_{\mathrm{I}}|\, e^{j\phi_{\mathrm{I}}}.$$

To illustrate the use of Fig. 9, assume for example that we are dealing with a system which has a phase of $\phi_{\mathrm{I}}=30°$ for the image impedance seen from the left-hand end, and of $\phi_{\mathrm{II}}=60°$ seen from the right-hand end. We deal then with the corresponding curves on Fig. 9, and the lower one on the left corresponds to $\log \rho_a$ and $\phi_a$, and the higher to $\log \rho_b$ and $\phi_b$. The envelope curve which takes the place of Fig. 6 for this case of image impedances of different phases is then constructed by finding the two ordinates $\log \rho_a$ and $\log \rho_b$, which correspond to the two angles $\phi_a$ and $\phi_b$, such

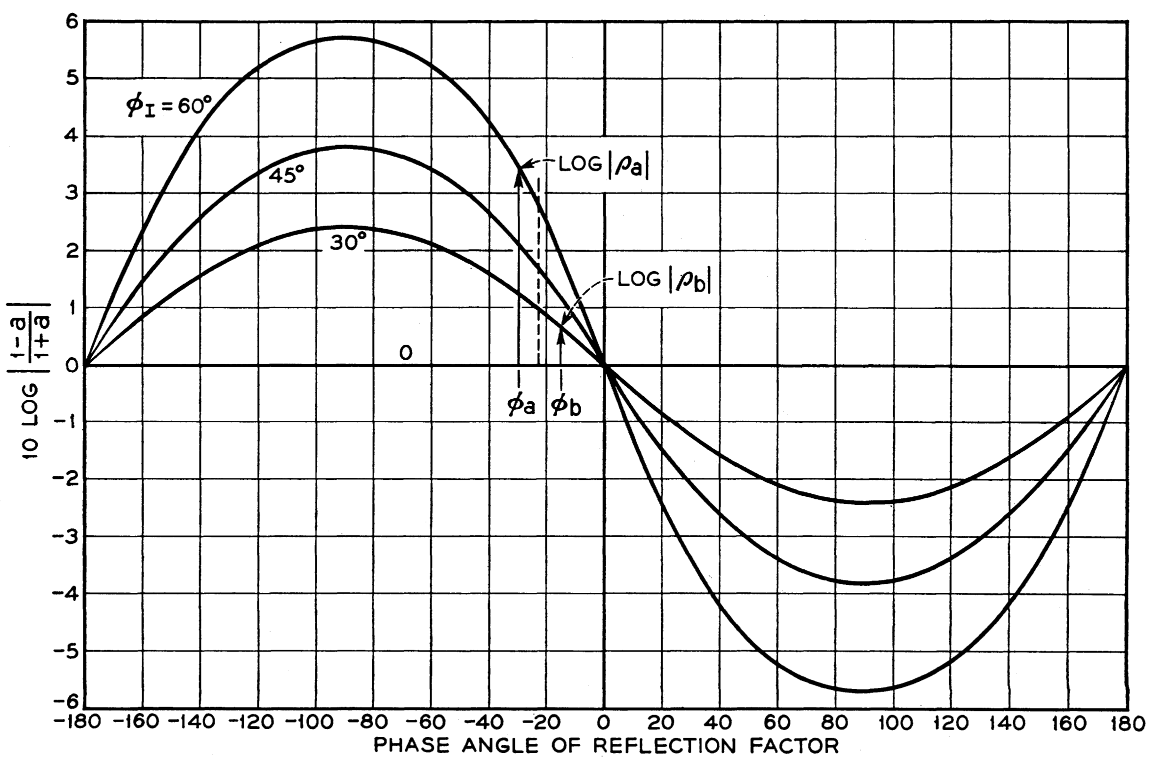


Fig. 9—Graph of logarithm of reflection factor

that the sum of the two ordinates is a maximum when $\phi_a+\phi_b$ equals the envelope angle $\psi$.

Algebraically this is equivalent to requiring:

$$y_a + y_b = \text{maximum when } x_a + x_b = \text{constant.}$$

Hence, for the maximum

$$\frac{dy_a}{dx_a}\,dx_a + \frac{dy_b}{dx_b}\,dx_b = 0.$$

But,

$$dx_b = -\,dx_a,$$

so that

$$\frac{dy_a}{dx_a} = \frac{dy_b}{dx_b}\cdot$$

This gives the clue to the graphic solution. Suppose that $\psi=-45°$. Starting then with $\phi_a$ and $\phi_b$, both equal to $-22.5°$, we note that the slope of the curve for $\phi_I=60°$ is much greater than that for $\phi_I=30°$. Consequently, we take a value for $\phi_a$ greater in magnitude than $-22.5°$, and for $\phi_b$ an equal amount less in magnitude than $-22.5°$ so that their sum again equals 45°. We note whether the slope of the $\phi_I=60°$ curve corresponding to $\phi_a$ still remains greater than that of the $\phi_I=30°$ curve corresponding to $\phi_b$. If so, the departures of $\phi_a$ and $\phi_b$ from the mean of $-22.5°$ should be increased further until the two slopes are equal. When such a pair of values has been found, the envelope of the product curve for $\psi=45°$ has the value

$$\log \rho_a + \log \rho_b$$

for its logarithm.

The process is tedious and would have to be repeated for each value $\psi$ of the envelope phase, and besides, the whole graphical construction would have to be repeated for each frequency in order to find the entire three-dimensional contour outside of which the loss must remain if stability is to be insured. Straightforward analytical solution does not offer much hope, either, for the difficulties appear to become even greater. It may happen, of course, that some change of variable or other algorithm will be found, but the probability is not at present very favorable.

In some cases, and particularly when attention is directed toward a general philosophical approach rather than to operating criteria, an extension of the stability conditions along the line proposed by Gewertz[3] has proved useful. In this extension of the work of Gewertz, the coefficients of (1) are written in the matrix form

$$\begin{vmatrix} R_{11}+jX_{11} & R_{12}+jX_{12} \\ R_{21}+jX_{21} & R_{22}+jX_{22} \end{vmatrix}.$$

It may then be shown by the argument given above, that the system is stable for any passive termination, providing that the following conditions are satisfied at all frequencies:

$$R_{11} > 0$$

$$R_{22} > 0$$

$$4(R_{11}R_{22} + X_{12}X_{21})(R_{11}R_{22} - R_{12}R_{21}) - (R_{12}X_{21} - R_{21}X_{12})^2 > 0.$$

In the event that $Z_{12}=Z_{21}$, it will readily be recognized that the last of these three relations reduces to the form

$$(R_{11}R_{22} - R_{12}^2) > 0,$$

which has come to be known as the Gewertz condition. On the other hand, in the event that $Z_{21}=-Z_{12}$ we have the alternative

$$(R_{11}R_{22} - X_{12}^2) > 0.$$

In the symmetrical case, where $Z_{11}=Z_{22}$, and where $Z_{12}=Z_{21}$, it may be shown that the Gewertz condition becomes

$$R_{11} - R_{12} > 0,$$

whence it follows that the real part of

$$Z_I\,\frac{1-\sqrt{\Gamma_0}}{1+\sqrt{\Gamma_0}}$$

must be greater than zero. This means that the resistive component of the short-circuit impedance of a hypothetical network of half the electrical length of the actual network, should always be positive. From this it is easy to deduce the relationship

$$(1 - |\Gamma_0|)\cos\phi - 2\sqrt{|\Gamma_0|}\sin\phi\sin\beta > 0,$$

where

$$Z_I = |Z_I|\,e^{j\phi}$$

$$\Gamma_0 = |\Gamma_0|\,e^{-2j\beta}.$$

From this equation it follows that, in case $\sin\beta=1$, we have

$$|\Gamma_0| < \frac{1-|\sin\phi|}{1+|\sin\phi|},$$

which agrees with the results previously attained, and shows the connection between the two methods of approach, namely the consideration of the matrix components on the one hand, and of the image parameters on the other.

It is important to point out that, in the present extended form of the Gewertz relations, all that is assured is that the network shall remain passive regardless of what passive terminations are attached. It does not follow that the network has all of the properties of a passive system, in the sense that it may be imbedded in a general network system involving passive feedback from the output to the input and still remain completely stable. A system composed entirely of passive ele-

[3] C. M. Gewertz, "Network Synthesis," The Waverly Press, 1933; pp. 45–63.

ments would, of course, be stable under these conditions.

Notwithstanding the difficulties of the general case where the image impedances at the two ends of the system are different from each other, some of the conclusions which have been pointed out are of broad validity, and in the more restricted case where the two image impedances are the same, quantitative results may be computed with moderate ease for any phase angle of the image impedances. Expressed in terms of the image gain and the image impedances, the relations are so important and general that a few examples may make their meaning clearer.

For the first one, the unilateral amplifier will be considered in order to show that, even in this case, the general principles and method of analysis apply. The schematic circuit diagram is shown in Fig. 10, and the four-pole equations are given on the figure. It is important to note that a feedback admittance $Y_x$ is included for purposes of analysis, but that this is ultimately allowed to become so small that feedback disappears.

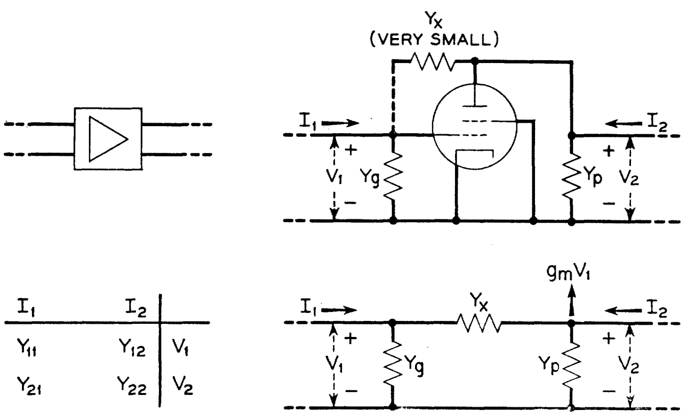


Fig. 10—Development of unilateral amplifier.

It happens that admittances are more convenient than impedances to deal with here, but the form of the various relations is not changed thereby. Application of (14), (15), and (22), gives the image parameters $Y_{\text{I}}$, $Y_{\text{II}}$, and $\Gamma_0$ as follows:

$$Y_{\text{I}} = (Y_g + Y_x)\sqrt{1 - \frac{Y_x(g_m - Y_x)}{(Y_g + Y_x)(Y_p + Y_x)}}$$

$$Y_{\text{II}} = (Y_p + Y_x)\sqrt{1 - \frac{Y_x(g_m - Y_x)}{(Y_g + Y_x)(Y_p + Y_x)}}$$

$$\Gamma_0 = \frac{1 - \sqrt{1 - \dfrac{Y_x(g_m - Y_x)}{(Y_g + Y_x)(Y_p + Y_x)}}}{1 + \sqrt{1 - \dfrac{Y_x(g_m - Y_x)}{(Y_g + Y_x)(Y_p + Y_x)}}}.$$

As the feedback admittance $Y_x$ is allowed to become very small, (impedance very high), the image admittances easily and gracefully approach the values $Y_g$ and $Y_p$, respectively. At the lower frequencies before transit-time effects enter, these are ordinary passive admittances. The image gain approaches zero, but the way it does this can best be seen by using the binomial theorem to expand the radical in the numerator. This gives

$$\Gamma_0 \rightarrow \frac{1}{4}\,\frac{Y_x(g_m - Y_x)}{(Y_g + Y_x)(Y_p + Y_x)} \rightarrow \frac{1}{4}\,\frac{g_m Y_x}{Y_g Y_p}. \tag{34}$$

In this form, it can be seen from (24) that the operating gain from left to right with matched terminations becomes

$$\Gamma_{21}' \doteq \frac{g_m - Y_x}{Y_x}\cdot\frac{1}{4}\,\frac{g_m Y_x}{Y_g Y_p} \rightarrow \frac{1}{4}\,\frac{g_m^2}{Y_g Y_p}, \tag{35}$$

which is recognizable as the conventional expression for this gain. On the other hand, $\Gamma_0$ itself approaches zero, and the matched gain $\Gamma_{12}'$ from right to left likewise approaches zero, and in such a way that the image gain $\Gamma_0$ is the geometric mean between $\Gamma_{12}'$ and $\Gamma_{21}'$. Whenever the feedback admittance $Y_x$ is not quite zero, the circuit may yet be stable for all passive impedance terminations, but only providing that the image gain $\Gamma_0$, multiplied by the reflection coefficients, does not encircle the point $(1, j0)$.

This rather extreme illustration was chosen first to demonstrate the generality of the analysis, and to show how it applies in the unilateral case.

As an example of the bilateral case, the properties of the 22-type repeater will be considered. Fig. 1 shows the general schematic and, when the impedances seen by the hybrid coils on the network side, the transmitting side, and the receiving side are completely balanced, the image impedances are equal to the impedance of the passive balancing networks and are independent of the repeater gain. When the over-all image impedance of a system containing 22-type repeaters is a pure resistance, the repeaters may be adjusted until the gain of the system approaches unity before singing can take place. In the more usual case, the image impedances of the individual repeaters are adjusted to match that of the connecting line, which has an appreciable phase angle. Consequently the gain of the system must be held to a flat value of

$$\frac{1 - |\sin\phi_{\text{I}}|}{1 + |\sin\phi_{\text{I}}|},$$

or else must be tailored to fit the conditions discussed in connection with Fig. 6. However, the expedient of providing initial and terminating repeaters, whose input and output hybrids are matched to a pure resistance, will allow the system gain to be brought up to unity even in this case. With ideal impedance matches, the margin which must be allowed in practical design then depends upon the variations in repeater gains and line losses under operating conditions, and not upon the number of sections in the system or upon the over-all line loss. Extra margins are required for unavoidable impedance mismatches resulting from line irregularities.

With other types of repeaters, such as the 21-type illustrated in Fig. 11, the image gain and the image impedances are not so easy to adjust independently. However, there seems to be nothing fundamental to prevent supplying the terminals of the system with networks to provide a purely resistive image impedance. Practically, there are many cases where the construction of such networks offer excessive complexity, though in others their use may be quite feasible. The gain of the system can then be made unity before singing will occur, with any possible combination of passive impedances attached to the terminals.

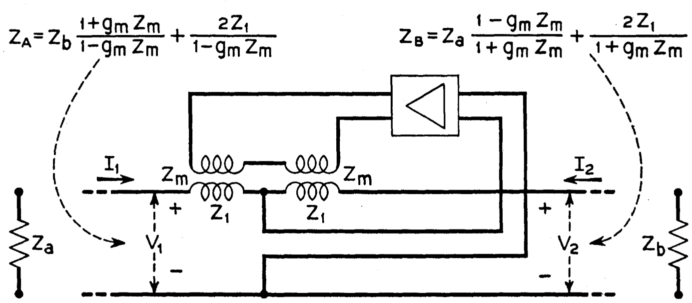


Fig. 11—Schematic of 21-type repeater.

This brings out the point that it is perfectly possible to have a system that is quite stable when terminated at mid-section points, but may be quite unstable when terminated at half-load points, and vice versa. This can happen, for example, when the image impedance is a pure resistance in the one case, but not in the other. It can happen more generally, however, whenever the phase of the image impedance is different in the two cases, and consequently the allowable gain is different.

The analysis also shows that the attempt to improve the singing margin of a system by the addition of pads of various kinds is quite futile in some cases but not in others. If a system has an image impedance that is a pure resistance, its gain may be made unity before singing can occur. The addition of a resistive pad will allow the repeater gains to be increased until the over-all gain is gain unity, but no more, and the system is back again where it started. A reactive pad added to such a system will do harm because it will produce image impedances with reactive components, and the over-all gain must remain correspondingly less than unity. However, as shown before, a phase-correcting network, or pad, applied to a system having an appreciable phase angle for its image impedance, will be helpful. We are thus led to the conclusion that an advantageous terminating four-pole is one that transforms the image impedance into a pure resistance in those cases where it initially has a reactive component.

This observation also gives the key to the best design objective for repeatered systems in general. That objective is to cause the image impedance of the system to be purely resistive to as close a degree as possible, while bringing the gain as nearly to unity as is consistent with safe singing margins.

These examples also illustrate a general conclusion that may be stated as follows:

The external stability of all systems depends only upon the phase of the image impedances and magnitude of the image gain, and not at all upon the details of the internal arrangements of the system by which these quantities are attained.

It does not follow, however, that all systems are alike in terms of the percentage change of voltage on the vacuum tubes which provide the repeater gain or the negative impedance loading, or in terms of the complexity of the equalizing and phase-correcting networks required to give the desired image-impedance terminations. The image gain of a 22-type repeater without feedback is proportional almost directly to the effective voltage of the dc supply source, while the image impedances are almost independent of this voltage. The image gain of a line with negative impedance loading may, under some conditions, vary much less rapidly with supply voltage to the tubes that furnish the negative impedance. Also, systems vary greatly in the amount of trouble resulting from line impedance irregularities.

Consequently, rather than regarding the theory here presented as saying that all systems having the same image parameters will behave alike, it may be more useful to turn the statement around and regard the theory as saying what has to be done to a given system in order that it shall be capable of operating as well as some other system. Conversely, the theory also tells how much more loss the given system perforce must have than a reference system in order to remain unconditionally stable, and it sets up specific and definite standards for the reference system. The present paper has stressed the applicability of the image parameter concept to the determination of singing conditions in telephone systems. However, the methods developed are also capable of dealing with such other properties as talker and listener echo, which are equally important in some applications. These have not been discussed in detail because the paper already is fairly long and because, with the fundamental background as presented, the reader is in a position to carry out a number of extensions for himself.

As a closing word, a few remarks concerning bibliography references are in order. It will be noticed that very few occur in the text. This is because the writer is aware of very few that have a specific and direct bearing on the mode of development of the subject which was employed. He wishes however to express appreciation of the helpful and stimulating conversations he has had with many of his colleagues on the technical staff of the Bell Telephone Laboratories. As general background to the use of image parameters in active circuit analysis, the following may be mentioned in addition to the standard modern text books:

1. H. A. Wheeler, "Wide-band amplifiers for television," Proc. I.R.E., vol. 27, pp. 429–438; July, 1939.
2. A. J. Ferguson, "Termination effects in feedback amplifier chains," *Canad. Jour. Phys.*, Section A, vol. 24, pp. 56–278; July, 1946.