V. CONCLUSIONS

We have introduced and mathematically defined the concept of network-element-value solvability, the determination of network-element values in terms of external behavior. Conditions for solvability of linear, passive, lumped-parameter networks with possible internal energy sources have been obtained, relating the number of elements capable of being evaluated to the number of available and partly available terminals.

More work is required in the following areas:

1) Devise general methods of determining sufficient conditions for network solvability in terms of specific network topology.
2) Determine computer programmable algorithms for network element evaluation for solvable networks.
3) Develop methods of handling unilateral circuit elements.
4) Minimize the effects of measurement errors by optimization of measurement procedures.

Stability and Power-Gain Invariants of Linear Twoports*

J. M. ROLLETT†, MEMBER, IRE

Summary—It is shown that the stability of a linear twoport is invariant under arbitrary lossless terminations, under interchange of input and output, and under "immittance substitution," a transformation group involving the arbitrary interchanging of impedance and admittance formulations at both ports. The quantity

$$k = \frac{2\,Re\,(\gamma_{11})\,Re\,(\gamma_{22}) - Re\,(\gamma_{12}\gamma_{21})}{|\,\gamma_{12}\gamma_{21}\,|}$$

(where the γ may be any of the conventional immittance z, y, or hybrid h, g matrix parameters) is the simplest invariant under these transformations, and describes uniquely the degree of stability, provided $Re(\gamma_{11})$, $Re(\gamma_{22}) \geq 0$; the larger k is, the greater the stability, and in particular $k = 1$ defines the boundary between unconditional and conditional stability. The quantity k is thus the basic invariant stability factor. Its definition is also extended to include the effect of terminating immittances, which may be padding resistances or source and load immittances, or both.

Certain power-gain functions, including the maximum available power gain, are shown to be invariant under immittance substitution, and k is identified as a function of ratios between them, where they exist. This provides a fundamental way of determining k, apart from calculating it from matrix parameters, and indicates that it is a measure of an inherent physical property.

INTRODUCTION

TWO centers of interest in modern circuit theory are the passivity and stability of linear networks.[1] The concept of the passivity (or activity) of linear twoports has been greatly illuminated by Mason's invariant U function,[2,3] but no similar invariant has so far been proposed for stability. The present work (suggested by the analogy between the two concepts) shows that there is an invariant stability factor which uniquely characterizes the degree of conditional or unconditional stability of a linear twoport, much as Mason's invariant U function uniquely characterizes its passivity or activity.

The stability of linear twoports has been discussed by many authors.[1,4] A measure of stability, including the effect of source and load immittances, was first proposed by Stern[5,6] and Bahrs,[7,8] and has been discussed by Venkateswaran and Boothroyd.[9] The invariant stability factor introduced here is, in its basic form, identical with

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† British Telecommunications Research Ltd., Taplow Court, Taplow, near Maidenhead, Berks., England.

[1] An introduction and bibliography are given by E. F. Bolinder, "Survey of some properties of linear networks," IRE TRANS. ON CIRCUIT THEORY, vol. CT-4, pp. 70–78; September, 1957.

[2] S. J. Mason, "Power Gain in Feedback Amplifiers," Res. Lab. of Electronics, Mass. Inst. Tech., Cambridge, Tech. Rept. 257, August 25, 1953; IRE TRANS. ON CIRCUIT THEORY, vol. CT-1, no. 2, pp. 20–25, June, 1954.

[3] S. J. Mason, "Some properties of three-terminal devices," IRE TRANS. ON CIRCUIT THEORY, vol. CT-4, pp. 330–332; December, 1957.

[4] F. B. Llewellyn, "Some fundamental properties of transmission systems," PROC. IRE, vol. 40, pp. 271–283; March, 1952.

[5] A. P. Stern, "Considerations on the stability of active elements and applications to transistors," 1956 IRE NATIONAL CONVENTION RECORD, pt. 2, pp. 46–52.

[6] A. P. Stern, "Stability and power gain of tuned transistor amplifiers," PROC. IRE, vol. 45, pp. 335–343; March, 1957.

[7] G. S. Bahrs, "Amplifiers Emplying Potentially Unstable Elements," Electronics Res. Lab., Stanford University, Stanford, Calif., Tech. Rept. 105; May 7, 1956.

[8] G. S. Bahrs, "Stable amplifiers employing potentially unstable transistors," 1957 IRE WESCON CONVENTION RECORD, pt. 2, pp. 185–189.

[9] S. Venkateswaran and A. R. Boothroyd, "Power gain and bandwidth of tuned transistor amplifier stages," *Proc. IEE*, vol. 106B, Suppl. no. 15, pp. 518–529; 1959.

a quantity first defined by Aurell,[10] without any indication of its possible significance, and is the inverse of a "criticalness factor" defined by Linvill and Schimpf.[11] It differs from Stern's in that it has only one unique value (a property half recognised by Aurell but not mentioned by Linvill and Schimpf), whereas Stern's takes on four different values, depending on the formulation used.[9]

The Stability Transformation Group

A linear twoport is unconditionally stable if, with arbitrary passive terminations, its characteristic frequencies remain in the left half of the complex frequency plane. An equivalent statement is that the real part of the immittance looking in at only one of the two ports remains positive with arbitrary passive terminations at the other, provided also that the characteristic frequencies of the twoport with ideal terminations (infinite immittances, *i.e.*, open or short circuits, as appropriate) lie in the left half-plane. This last condition will be assumed to hold in what follows.

The stability of a twoport increases with lossy terminations, but remains constant with lossless terminations. Thus any quantity which indicates conditional or unconditional stability must (if it exists) be invariant under arbitrary lossless terminations. Furthermore, since (with the proviso above) stability depends on the positive realness of the immittance looking in at either port, it must be invariant under interchange of input and output; and since the immittance may be expressed either as impedance or admittance, it must be invariant under the interchange of impedance and admittance formulations at either port. The arbitrary interchanging of impedance and admittance formulations is carried out most easily by substituting any one set of the conventional impedance, admittance or hybrid matrix[12] parameters (z, y, h, g) for any other, and it is easy to show that the substitution operators form a group (isomorphous with the "vierergruppe"). This transformation group will be called *immittance substitution.*

The quantity characteristic of stability we are looking for is therefore invariant under arbitrary lossless terminations, under interchange of input and output, and under immittance substitution. These transformations may be combined to form a single infinite group, which is the "direct product" of the three separate groups, and contains all the transformations associated with stability.

The Invariant Stability Factor

The well-known criterion for a linear twoport to be unconditionally stable[1,4] is, in admittance parameters,

$$2g_{11}g_{22} \geq |y_{12}y_{21}| + Re\,(y_{12}y_{21}), \tag{1}$$

provided

$$g_{11},\ g_{22} \geq 0$$

where $g_{11} = Re\ (y_{11})$, etc. Any quantity indicative of conditional or unconditional stability must include all the information contained in (1), which is clearly invariant under arbitrary lossless terminations and under interchange of input and output. Now it can be shown that the quantity k, defined by

$$k \triangleq \frac{2g_{11}g_{22} - Re\,(y_{12}y_{21})}{|y_{12}y_{21}|}, \tag{2}$$

is invariant also under immittance substitution. It is therefore invariant under the complete group of transformations associated with stability. As it is the simplest such invariant, to which all others are trivially related (in the mathematical sense), it will be called the *invariant stability factor*. Since the value of k remains unchanged when z, h or g parameters are substituted for y parameters, it is convenient to generalize the notation, and so γ will be used for any of the z, y, h, g twoport parameters. In this notation

$$k \triangleq \frac{2\rho_{11}\rho_{22} - Re\,(\gamma_{12}\gamma_{21})}{|\gamma_{12}\gamma_{21}|}, \tag{3}$$

where $\rho_{11} = Re\ (\gamma_{11})$, etc. The value of k lies between $+\infty$ and -1, if ρ_{11}, $\rho_{22} \geq 0$.

The criterion for unconditional stability may now be written as

$$k \geq 1 \tag{4}$$

provided

$$\rho_{11},\ \rho_{22} \geq 0,$$

and this is the fundamental property of the invariant stability factor. Furthermore, when k is positive and large compared with unity, the degree of unconditional stability is high; when k is only just greater than unity, the twoport is near the boundary between unconditional and conditional stability defined by $k = 1$. When $1 > k \geq -1$, the twoport is in the region of conditional stability, and it is always possible to choose terminations which will lead to negative real input or output immittances or which will result in oscillations; that is, the characteristic frequencies can be located in the right half-plane or on the real frequency axis.

If external immittances Γ_1, Γ_2 are added to the twoport so that they may be regarded as part of it from the viewpoint of stability, the definition of the stability factor may be extended, and an over-all stability factor K can be defined by

$$K \triangleq \frac{2(\mathrm{P}_1 + \rho_{11})(\mathrm{P}_2 + \rho_{22}) - Re\,(\gamma_{12}\gamma_{21})}{|\gamma_{12}\gamma_{21}|}, \tag{5}$$

[10] C. G. Aurell, "Representation of the general linear four-terminal network and some of its properties," *Ericsson Tech.*, vol. 11, pp. 155–179; 1955.

[11] J. G. Linvill and L. G. Schimpf, "The design of tetrode transistor amplifiers," *Bell Sys. Tech. J.*, vol. 35, pp. 813–840; July, 1956.

[12] R. F. Shea, Ed., "Principles of Transistor Circuits," John Wiley and Sons, Inc., New York, N. Y.; 1953.

where $P_1 = Re\ (\Gamma_1)$, etc. The external immittances may be padding resistances or source and load immittances or both. The over-all stability factor K is useful in characterizing the stability of practical tuned amplifiers, working between source and load immittances whose real parts are known.

It is important to relate the invariant stability factor k, which so far has been introduced as a pure number, to other basic properties of the twoport. The next section shows how it is related to various invariant power gain functions, and indicates how it can in principle be determined, when it is positive, without knowing the matrix parameters. When it is negative, it can be determined indirectly by adding known padding resistances to make the overall stability factor positive,[13] or by calculation from matrix parameters.

Invariant Power-Gain Functions

There are three power-gain functions of interest which are invariant under immittance substitution and are directly related to the invariant stability factor. These are the maximum available power gain,[14] the maximum stable power gain,[15] and a function defined below as the minimum conjugate-termination transducer gain; each is introduced in turn.

Maximum Available Power Gain

The maximum available power gain[14] G_{MA} of a twoport is obtained when the input and output ports are simultaneously matched to their conjugate immittances. This is only possible if the device obeys the unconditional stability criterion (1). The expression for G_{MA} is

$$G_{MA} = \frac{|\gamma_{21}|^2}{2\rho_{11}\rho_{22} - Re\,(\gamma_{12}\gamma_{21}) + \sqrt{\{[2\rho_{11}\rho_{22} - Re\,(\gamma_{12}\gamma_{21})]^2 - |\gamma_{12}\gamma_{21}|^2\}}}, \tag{6}$$

or, in terms of the stability factor,

$$G_{MA} = \left|\frac{\gamma_{21}}{\gamma_{12}}\right| \frac{1}{k + \sqrt{(k^2 - 1)}} = \left|\frac{\gamma_{21}}{\gamma_{12}}\right| [k - \sqrt{(k^2 - 1)}]. \tag{7}$$

As expected on general physical grounds, G_{MA} is invariant under immittance substitution. Since the factor involving k is invariant, the other factor $|\gamma_{21}/\gamma_{12}|$ is also invariant. It is discussed and identified in the next section.

The optimum load immittances, which provide the simultaneous conjugate match, may be conveniently expressed using k. If they are denoted by Γ_{1opt} at the input and Γ_{2opt} at the output, then

$$\Gamma_{1opt} = \frac{|\gamma_{12}\gamma_{21}|\sqrt{(k^2 - 1)}}{2\rho_{22}} + j\left[\frac{Im\,(\gamma_{12}\gamma_{21})}{2\rho_{22}} - \sigma_{11}\right], \tag{8}$$

(where $\sigma_{11} = Im\ (\gamma_{11})$, etc.) and similarly for Γ_{2opt}. The total self immittances are particularly concise:

$$\Gamma_{1opt} + \gamma_{11} = \frac{\gamma_{12}\gamma_{21} + |\gamma_{12}\gamma_{21}|[k + \sqrt{(k^2 - 1)}]}{2\rho_{22}} \tag{9}$$

and similarly for $(\Gamma_{2opt} + \gamma_{22})$.

The optimum load immittances (8) are invariant under immittance substitution, as would be expected on general physical grounds. However, the total self immittances (9) are *not* invariant; that is, although the expressions for them are similar, using different matrix parameters, their physical values are different.

The expression for maximum available power gain (7) is made up of two factors, the one involving k being reciprocal (*i.e.*, invariant with respect to interchange of forward and reverse transfer parameters), while the other (which is discussed below) is nonreciprocal. Thus the factor $[k - \sqrt{(k^2 - 1)}]$ is the maximum efficiency[10,16] of the reciprocal part of the twoport, while its inverse $[k + \sqrt{(k^2 - 1)}]$ may be called the *minimum reciprocal attenuation*, provided the unconditional stability criterion holds. (It must be remembered that a device can be both reciprocal and active,[17] and that it then necessarily violates the unconditional stability criterion.) The minimum reciprocal attenuation is purely a function of k, and this gives an insight into the nature of the physical property of which k is a measure.

If the maximum available power gain in the reverse direction, found by interchanging the two ports, is denoted by G_{MA}^r, then we have

$$\frac{G_{MA}}{G_{MA}^r} = \left|\frac{\gamma_{21}}{\gamma_{12}}\right|^2 \tag{10}$$

and

$$\sqrt{(G_{MA}G_{MA}^r)} = k - \sqrt{(k^2 - 1)}, \tag{11}$$

which enable $|\gamma_{21}/\gamma_{12}|$ and k to be determined, provided $k > 1$ and $\rho_{11}, \rho_{22} > 0$.

Maximum Stable Power Gain

The maximum stable power gain[15] is defined as follows. If $k < 1$ or if ρ_{11} or $\rho_{22} < 0$, then lossy padding immittances

[13] Thus if P_a, P_b are placed successively at one port, and the corresponding (positive) over-all stability factors K_a, K_b are measured, then the basic stability factor is given by $K = (P_aK_b - P_bK_a)/(P_a - P_b)$.

[14] "IRE Standards on Electron Tubes," Proc. IRE, vol. 45, pp. 983–1010; July, 1957.

[15] M. A. Karp, "Power gain and stability," IRE Trans. on Circuit Theory, vol. CT-4, pp. 339–340; December, 1957.

[16] This quantity is discussed in a paper which has just been brought to the attention of the author: S. Venkateswaran, "An invariant stability factor and its physical significance," IEE Mono. No. 468E, September, 1961, to be republished in *Proc. IEE*, pt. C.

[17] J. Shekel, "Reciprocity relations in active 3-terminal elements," Proc. IRE, vol. 42, pp. 1268–1270; August, 1954.

Γ_1 and Γ_2 can always be placed at the two ports so as to make the real parts of the self parameters $(P_1 + \rho_{11})$, $(P_2 + \rho_{22}) > 0$, and the over-all stability factor $K \rightarrow 1$, *i.e.*, so that the over-all twoport approaches the boundary between unconditional and conditional stability. The maximum available power gain then tends towards its maximum stable value,

$$G_{MA} \rightarrow \left| \frac{\gamma_{21}}{\gamma_{12}} \right| \triangleq G_{MS}, \tag{12}$$

and G_{MS} is called the *maximum stable power gain.*[15] It is also invariant under immittance substitution.[9]

The use of lossy elements means that K for the padded twoport must be greater than k for the basic twoport. This implies that G_{MS} is only defined for devices where $k \leq 1$. However, it is useful to extend the definition of G_{MS} to include devices with $k > 1$, and this is done by observing that, in principle, negative resistances could be chosen for the terminations Γ_1, Γ_2, so as to allow K to be less than k. The stability boundary $K = 1$ can then again be approached, so that G_{MS} can be defined as the maximum stable power gain for all devices.

Having identified the quantity $| \gamma_{21}/\gamma_{12} |$, we can write, in general, for a device obeying the unconditional stability criterion,

$$G_{MA} = G_{MS}/[k + \sqrt{(k^2 - 1)}]; \tag{13}$$

i.e., the maximum available power gain is given by dividing the maximum stable power gain by the minimum reciprocal attenuation. The value of G_{MS} may be found as indicated in the previous section.

Minimum Conjugate-Termination Power Gain

Here we introduce a power-gain function which suggests a method for determining k over a wider range of values than the previous method (11) allows.

Consider the following sequence of operations. Conjugately match at one port, with the second arbitrarily terminated in Γ. Then remove the arbitrary termination Γ and conjugately match at the second port, thereby destroying the conjugate match at the first port (provided the device is nonunilateral). The transducer gain[14] is now a function of the termination Γ and has a minimum which occurs when Γ is lossless. This may be called the *minimum conjugate-termination transducer gain* G_{CT}, where

$$G_{CT} \triangleq \frac{| \gamma_{21} |^2}{4\rho_{11}\rho_{22} - 2 \, Re \, (\gamma_{12}\gamma_{21})}, \tag{14}$$

or, in terms of invariants already discussed,

$$G_{CT} = \left| \frac{\gamma_{21}}{\gamma_{12}} \right| \frac{1}{2k} = G_{MS}/2k. \tag{15}$$

This quantity has been introduced by Linvill and Schimpf,[11] who call it "P_{oo}/P_{io}", but make no mention of its extremum properties. Its importance in the present context lies in the fact that it enables us to determine k, except when $k \leq 0$. For, although the device is not unconditionally stable for $0 \leq k < 1$, it can easily be shown that when $\Gamma \rightarrow \infty$, the sequence of operations outlined above always leads to the immittances looking in at both ports having positive real parts.

Thus denoting by G_{CT}^r the reverse quantity, found by interchanging input and output, we have

$$k = 1/2\sqrt{(G_{CT}G_{CT}^r)} \tag{16}$$

for $k > 0$, and provided $\rho_{11}, \rho_{22} > 0$.

Discussion

An invariant stability factor has been introduced, together with its associated transformation group, and its essential properties described. Its relationships with certain invariant power-gain functions have been investigated, and shown to lead to basic methods of determining it, without calculating it from matrix parameters.

This suggests that k is a measure of a fundamental physical property, despite the fact that it can only be determined indirectly when it is negative.[18] Although k is the simplest stability invariant, to which all others are trivially related, in the mathematical sense, it may turn out that some function of k can be more closely identified with a physical property than k itself. However, its invariant properties ensure that no other quantity can convey more information about stability, and it is in this sense that k is unique.

The extension of the definition to include the effects of terminating immittances in the over-all stability factor K should prove useful in the design of a common class of amplifier, *i.e.*, those which are unneutralized, resistively mismatched and reactively tuned. In this connection it is worth pointing out that the transducer gain G_{TP}

$$G_{TR} = \frac{4P_1P_2 \, | \gamma_{21} |^2}{| (\Gamma_1 + \gamma_{11})(\Gamma_2 + \gamma_{22}) - \gamma_{12}\gamma_{21} |^2}, \tag{17}$$

(where Γ_1, Γ_2 are source and load immittances and $P_1 = Re \, (\Gamma_1)$, etc.) is also invariant under immittance substitution.[19] If a manageable relation between the overall K and G_T could be found, it would enable gain to be exchanged with stability on a quantitative basis; this has so far eluded the present author.

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[18] An exactly similar situation exists in the case of Mason's invariant U function.

[19] The ratio of forward to reverse transducer gain is the square of the maximum stable power gain, *cf.* (10). The derivation of this result in reference [17] appears to be fallacious.