Statistical Properties of a Sine Wave Plus Random Noise

By S. O. RICE

Introduction

IN SOME technical problems we are concerned with a current which consists of a sinusoidal component plus a random noise component. A number of statistical properties of such a current are given here. The present paper may be regarded as an extension of Section 3.10 of an earlier paper,[1] "Mathematical Analysis of Random Noise", where some of the simpler properties of a sine wave plus random noise are discussed.

The current in which we are interested may be written as

$$\begin{aligned} I &= Q\cos qt + I_N \\ &= R\cos(qt + \theta) \end{aligned} \tag{3.4}$$

where Q and q are constants, t is time, and I_N is a random (in the sense of Section 2.8 of Reference A) noise current. When the second expression involving the envelope R and the phase angle θ is used, the power spectrum of I_N is assumed to be confined to a relatively narrow band in the neighborhood of the sine wave frequency $f_q = q/(2\pi)$. This makes R and θ relatively slowly (usually) varying functions of time.

In Section 1, the probability density and cumulative distribution of I are discussed. In Section 2, the upward "crossings" of I (i.e., the expected number of times, per second, I increases through a given value I_1), are examined.

The probability density and the cumulative distribution of R are given in Section 3.10 of Reference A. The crossings of R are examined in Section 4 of the present paper.

The statistical properties of θ', the time derivative of the phase angle θ, are of interest because the instantaneous frequency of I may be defined to be $f_q + \theta'/(2\pi)$. The probability density of θ' is investigated in Section 5 and the crossings of θ' in Section 6. θ' is a function of time which behaves somewhat like a noise current and may accordingly be considered to consist of an infinite number of sinusoidal components. The problem of determining the "power spectrum" $W(f)$ of θ', i.e., the distribution of the mean square value of the components as a function of frequency, is attacked in

[1] *B.S.T.J.* 23 (1944), 282–332 and 24 (1945), 46–156. This paper will be called "Reference A".

Sections 7 and 8. The correlation function of θ' is expressed in terms of exponential integrals in Section 7, the power spectrum of I_N being assumed symmetrical and centered on f_q. In Section 8, values of $W(f)$ are obtained for the special case in which the power spectrum of I_N is centered on f_q and is of the normal-law type.

It is believed that some of the material presented here may find a use in the study of the effect of noise in frequency modulation systems. For example, the curves in Section 8 yield information regarding the noise power spectrum in the output of a primitive type of system. Also, the procedure employed to obtain the expression (5.7) for $\bar{\theta}'$ may be used to show that if

$$Q\cos[(A/\omega_0)\cos\omega_0 t + qt] + I_N = R\cos(qt + \theta)$$

the sinusoidal component of $d\theta/dt$ is[2]

$$-A\,(1 - e^{-\rho})\sin\omega_0 t$$

where ρ is the ratio $Q^2/(2\,\overline{I_N^2})$. This illustrates the "crowding effect" of the noise. The statistical analysis associated with R and θ of equations (3.4) (when the sine wave is absent) is similar to that used in the examination of the current returned to the sending end of a transmission line by reflections from many small irregularities distributed along the line. This suggests another application of the results.

Acknowledgment

I am indebted to a number of my associates for helpful discussions on the questions studied here. In particular, I wish to thank Mr. H. E. Curtis for his suggestions regarding this subject. As in Reference A, all of the computations for the curves and tables have been performed by Miss M. Darville. This work has been quite heavy and I gratefully acknowledge her contribution to this paper.

1. Probability Distribution of a Sine Wave Plus Random Noise

A current consisting of a sine wave plus random noise may be represented as

$$I = Q\cos qt + I_N \tag{1.1}$$

where Q and q are constants, t is the time, and I_N is a random noise current. The frequency, in cycles per second, of the sine wave is $f_q = q/(2\pi)$. In all

[2] The first person to obtain this result was, I believe, W. R. Young who gave it in an unpublished memorandum written early in 1945. He took the output of a frequency modulation limiter and discriminator to be proportional to either the signal frequency or to the instantaneous frequency (assumed to be distributed uniformly over the input band) of I_N according to whether Q is greater or less than the envelope of I_N. His memorandum also contains results which agree well with several obtained in this paper.

our work we denote the power spectrum of I_N by $w(f)$ and its correlation function by $\psi(\tau)$. The mean square value of I_N is denoted by ψ_0.

The study of the probability distribution of I is essentially a study of the integral[3]

$$p(I) = \frac{1}{\pi\sqrt{\psi_0}} \int_0^\pi \varphi\left[\frac{I - Q\cos\theta}{\sqrt{\psi_0}}\right] d\theta \tag{1.2}$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{1.3}$$

and $p(I)$ is the probability density of I, i.e. $p(I)dI$ is the probability that a value of current selected at random will lie in the interval $I, I + dI$. Another expression for $p(I)$ is given by equation (3.10–6) of Reference A, namely

$$p(I) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-izI-\psi_0 z^2/2} J_0(Qz)\, dz \tag{1.4}$$

where $J_0(Qz)$ denotes the Bessel function of order zero.

The substitutions

$$y = \frac{I}{\sqrt{\psi_0}}; \qquad a = \frac{Q}{\sqrt{\psi_0}} \tag{1.5}$$

enable us to write (1.2) as

$$p_1(y) = \sqrt{\psi_0}\, p(I) = \frac{1}{\pi} \int_0^\pi \varphi(y - a\cos\theta)\, d\theta, \tag{1.6}$$

where $p_1(y)$ denotes the probability density of y. This is the expression actually studied. Curves showing $p_1(y)$ and the cumulative distribution function

$$\begin{aligned} \int_{-\infty}^{I} p(I_1)dI_1 &= \int_{-\infty}^{y} p_1(y_1)dy_1 \\ &= \frac{1}{\pi}\int_0^\pi \varphi_{-1}(y - a\cos\theta)\, d\theta, \end{aligned} \tag{1.7}$$

where

$$\varphi_{-1}(x) = \int_{-\infty}^{x} \varphi(x_1)dx_1 = \tfrac{1}{2} + \tfrac{1}{2}\, erf\,(x/\sqrt{2}) \tag{1.8}$$

[3] W. R. Bennett, "Response of a Linear Rectifier to Signal and Noise," *Jour. Acous. Soc. Amer.* Vol. 15 (1944), 164–172, and *B.S.T.J.* Vol. 23 (1944), 97–113.

are shown in Figs. 1 and 2. The curves for $a = 10$ and $a = \sqrt{10}$ were computed by Simpson's rule from (1.6) and (1.7), and the curves for $a = 1$ were computed from the series (1.10) given below. Since both $\varphi(x)$ and

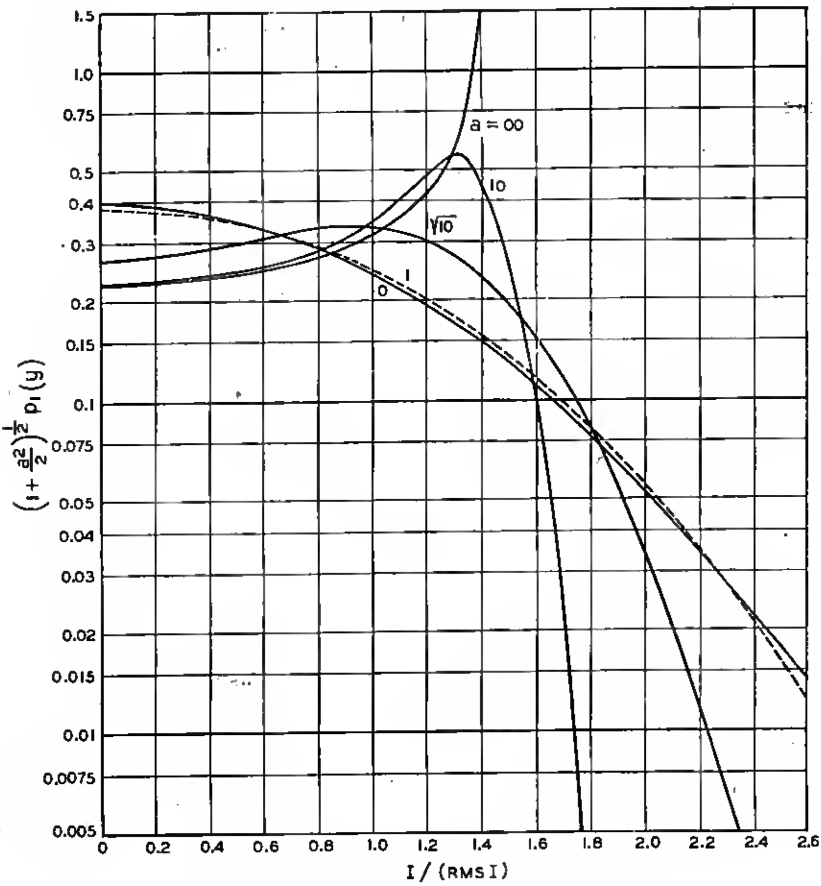

Fig. 1—Probability density of sine wave plus noise.

$I = Q\cos qt + I_N$, $a = Q/\sqrt{\psi_0}$, $y = I/\sqrt{\psi_0}$, $\sqrt{\psi_0} = \text{rms } I_N$
$p_1(y)\, dI/\sqrt{\psi_0}$ = probability total current lies between I and $I + dI$
$y(1 + a^2/2)^{-1/2} = I/(\text{rms } I)$. Curves are symmetrical about $y = 0$.

$\varphi_{-1}(x)$ are tabulated[4] functions the integrals in (1.6) and (1.7) are well suited to numerical evaluation.

[4] $\varphi(x)$ is given directly and $\varphi_{-1}(x)$ may be readily obtained from W.P.A., "Tables of Probability Functions," Vol. II, New York (1942). The functions $\varphi^{(m)}(y)$ are tabulated in Table V of "Probability and its Engineering Uses" by T. C. Fry (D. Van Nostrand Co., 1928).

The form assumed by $p_1(y)$ as the parameter a becomes large is examined in the latter portion (from equation (1.12) onwards) of the section.

Series which converge for all values of a but which are especially suited for calculation when $a \leq 1$ may be obtained by inserting the Taylor's series (in powers of x) for $\varphi(y + x)$ and $\varphi_{-1}(y + x)$, $x = -a \cos \theta$, in (1.6) and (1.7) and integrating termwise. When we introduce the notation[4]

$$\varphi^{(m)}(y) = \frac{d^m}{dy^m}\varphi(y) = \frac{1}{\sqrt{2\pi}}\frac{d^m}{dy^m}e^{-y^2/2} \tag{1.9}$$

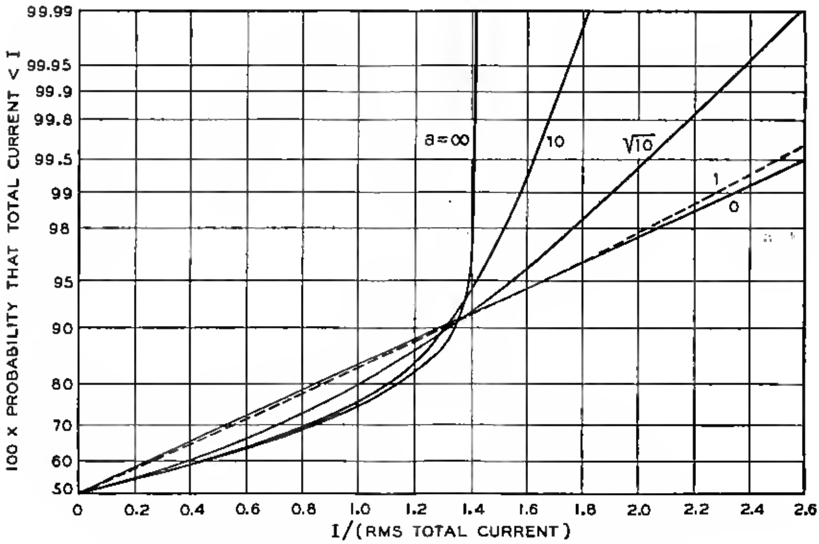

Fig. 2—Cumulative distribution of sine wave plus noise.

Ordinate $= 100 \int_{-\infty}^{y} p_1(y_1)\, dy_1$. See Fig. 1 for notation.

we obtain

$$\begin{aligned} p_1(y) &= \sum_{n=0}^{\infty} \frac{1}{n!n!}\left(\frac{a}{2}\right)^{2n} \varphi^{(2n)}(y) \\ \int_{-\infty}^{y} p_1(y_1)dy_1 &= \varphi_{-1}(y) + \sum_{n=1}^{\infty} \frac{1}{n!n!}\left(\frac{a}{2}\right)^{2n} \varphi^{(2n-1)}(y) \end{aligned} \tag{1.10}$$

The second equation of (1.10) may be shown to be valid by breaking the interval $(-\infty, y)$ into $(-\infty, 0)$ and $(0, y)$. In the first part,

$$\int_{-\infty}^{0} p_1(y_1)\, dy_1 = \varphi_{-1}(0)$$

since both sides have the value 1/2. In the second, term by term integration is valid since the series integrated are uniformly convergent as may be seen from the inequality

$$|\varphi^{(n)}(y)| \leq \left(\frac{n!}{2\pi}\right)^{1/2} \left(\frac{2}{\pi n}\right)^{1/4} [1 + 0(n^{-1}) + 0(y^2 n^{-1})], \qquad (1.11)$$

in which we suppose that y remains finite as $n \to \infty$. This may be obtained by using the known behavior of Hermite polynomials of large order.[5]

When $Q \gg rms\ I_n$ so that a is very large the distribution approaches that of a sine wave, namely

$$p_1(y) \sim \begin{cases} 0, & |y| > a \\ (a^2 - y^2)^{-1/2}/\pi, & |y| < a \end{cases}$$

$$\int_{-\infty}^{y} p_1(y_1)dy_1 \sim \frac{1}{2} + \frac{1}{\pi} \arcsin \frac{y}{a}, \qquad |y| < a \qquad (1.12)$$

In order to study the manner in which the limiting expressions (1.12) are approached it is convenient to make the change of variable

$$x = y - a \cos \theta, \qquad d\theta = [a^2 - (y - x)^2]^{-1/2}\, dx$$
$$z = x - y + a$$

in (1.6). We obtain

$$\begin{aligned} p_1(y) &= \frac{1}{\pi} \int_{y-a}^{y+a} \varphi(x)\, [a^2 - (y - x)^2]^{-1/2}\, dx \\ &= \frac{1}{\pi} \int_0^{2a} \varphi(z + y - a)[z(2a - z)]^{-1/2}\, dz. \end{aligned} \qquad (1.13)$$

An asymptotic (as a becomes large) expression for $p_1(y)$ suitable for the middle portion of the distribution where $a - |y| \gg 1$ may be obtained from the first integral in (1.13). Since the principal contributions to the value of the integral come from the region around $x = 0$ we are led to expand the radical in powers of x and integrate termwise. Legendre polynomials enter naturally since they are sometimes defined as the coefficients in such an expansion. Replacing the limits of integration $y + a$ and $y - a$ by $+\infty$ and $-\infty$, respectively and integrating termwise gives

$$\begin{aligned} p_1(y) &\sim \frac{(a^2 - y^2)^{-1/2}}{\pi} \left[1 + \sum_{n=1}^{\infty} (-)^n \frac{1.3.5.\,(2n - 1)}{(a^2 - y^2)^n} P_{2n}(it^{1/2}) \right] \\ &= \frac{(a^2 - y^2)^{-1/2}}{\pi} \left[1 + \frac{3t + 1}{2(a^2 - y^2)} + \frac{3(35t^2 + 30t + 3)}{8(a^2 - y^2)^2} + \cdots \right] \end{aligned} \qquad (1.14)$$

[5] A suitable asymptotic formula is given in Orthogonal Polynomials, by G. Szegö, *Am. Math. Soc. Colloquium, Pub.*, Vol. 23, (1939), p. 195.

where $t = y^2/(a^2 - y^2)$ and $P_{2n}(\ \ )$ denotes the Legendre polynomial of order $2n$. We have written this as an asymptotic expansion because it obviously is one when y, and hence t, is zero in which case

$$P_{2n}(0) = (-)^n \frac{1.3.5\cdots(2n-1)}{2.4.\cdots 2n}$$

When y is near a or is greater than a, a suitable asymptotic expansion may be obtained from the second integral in (1.13) by expanding $(2a - z)^{-1/2}$ in powers of $z/(2a)$ and integrating termwise. The upper limit of integration, $2a$, may be replaced by ∞ since $\varphi(z + y - a)$ may be assumed to be negligibly small when z exceeds 2a. We thus obtain

$$\begin{aligned} p_1(y) &\sim \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!}\left(\frac{1}{2a}\right)^{n+1/2} \int_0^\infty \varphi(z + y - a) z^{n-1/2}\, dz \\ &= \frac{\varphi(y-a)}{\pi} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!}\left(\frac{1}{2a}\right)^{n+1/2} \int_0^\infty e^{-z(y-a)-(z^2/2)} z^{n-1/2}\, dz \end{aligned} \tag{1.15}$$

where we have used the notation

$$(\alpha)_0 = 1, \qquad (\alpha)_n = \alpha(\alpha+1)\ \cdots\ (\alpha + n - 1).$$

The integrals occurring in (1.15) are related to the parabolic cylinder function[6] $D_m(x)$. Their properties may be obtained from the known properties of these functions or may be obtained by working directly with the integrals.

Suppose now that a is very large so that only the leading term in the series (1.15) for $p_1(y)$ need be retained.

Then

$$p_1(y) \sim a^{-1/2} F(y - a) \tag{1.16}$$

where

$$F(s) = \pi^{-1} 2^{-1/2} \int_0^\infty \varphi(z + s) z^{-1/2}\, dz \tag{1.17}$$

By writing out $\varphi(z + s)$, expanding $exp\ (-zs)$ in a power series, and integrating termwise we see that

$$\begin{aligned} F(s) &= \frac{\varphi(s)2^{-5/4}}{\pi} \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{\ell}{2} + \frac{1}{4}\right)}{\ell!} (-s\sqrt{2})^\ell \\ &= (2\pi)^{-1} s^{1/2} \varphi(s/\sqrt{2}) K_{\frac{1}{4}}(s^2/4) \end{aligned} \tag{1.18}$$

where K denotes a modified Bessel function.[7] The relation (1.18) may also

[6] Whittaker and Watson, "Modern Analysis," 4th ed. (1927), 347–351.

[7] A table of $K_{\frac{1}{4}}(x)$ is given by H. Carsten and N. McKerrow, Phil. Mag. S7, Vol. 35 (1944), 812–818.

be obtained from pair 923.1 of "Fourier Integrals for Practical Applications," by G. A. Campbell and R. M. Foster.[8]

A curve showing $F(y - a)$ plotted as a function of $y - a$ is given in Fig. 3. It was obtained from the relation

$$F(s) = 2^{1/4}\pi^{-3/2}\chi(-s/\sqrt{2})$$

where

$$\chi(x) = \int_0^\infty e^{-(x-w^2)^2}\,dw$$

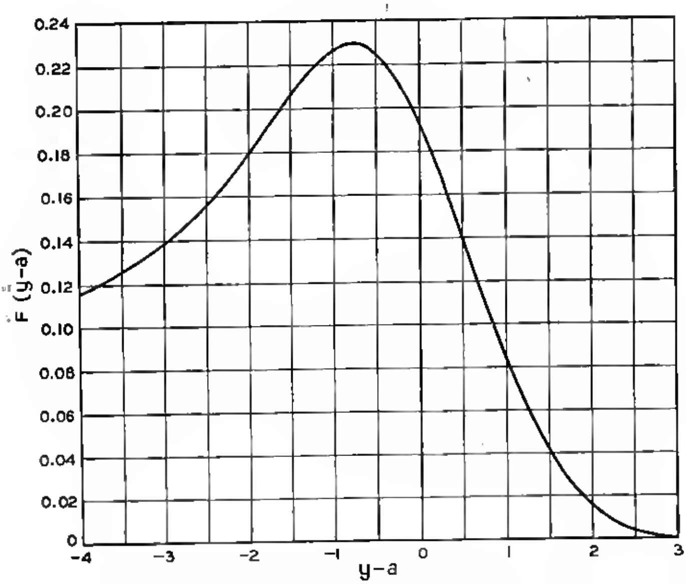

Fig. 3—Probability density of sine wave plus noise.
When rms $I_N << Q$ and I is near Q, $p_1(y) \sim a^{-1/2}F(y - a)$, $y - a = (I - Q)/(\text{rms } I_N)$.
See Fig. 1 for notation.

This function has been tabulated by Hartree and Johnston.[9]

The probability that I exceeds Q, or that y exceeds a, is, integrating the second of expressions (1.13),

$$\int_a^\infty p_1(y)\,dy = \frac{1}{\pi}\int_0^{2a}\frac{dz}{\sqrt{z(2a - z)}}\int_z^\infty \varphi(x)\,dx.$$

An asymptotic expansion may be obtained by expanding $(2a - z)^{-1/2}$ as in the derivation of (1.15) but we shall be content with the leading term.

[8] *Bell Telephone System Monograph* B-584.
[9] *Manchester Lit. and Phil. Soc. Memoirs*, v. 83, 183–188, Aug., 1939.

Using

$$\int_0^\infty z^{-1/2}\, dz \int_z^\infty \varphi(x)\, dz = \int_0^\infty \varphi(x)\, dx \int_0^x z^{-1/2}\, dz = 2^{1/4}\pi^{-1/2}\Gamma(\tfrac{3}{4})$$

we obtain

$$\int_a^\infty p_1(y)\, dy \sim 2^{-1/4}\pi^{-3/2}\Gamma(\tfrac{3}{4})a^{-1/2} = 0.185\cdots a^{-1/2} \tag{1.19}$$

For use in computations we list the following values

$$\Gamma(\tfrac{1}{4}) = 3.62561, \qquad \Gamma(\tfrac{3}{4}) = 1.22542, \qquad \Gamma(\tfrac{5}{4}) = 0.90640$$

2. Expected Number of Crossings of I per Second

In this section, we shall study two questions. First, what is the probability $P(I_1, t_1)dt$ of I increasing through the value I_1 (i.e. of I passing through the value I_1 with positive slope) during the infinitesimal interval $t_1, t_1 + dt$? Second, what is the expected number $N(I_1)$ of times per second I increases through the value I_1. When I_1 is zero, 2N(0) is the expected number of zeros per second, and when I_1 is large $N(I_1)$ is approximately equal to the expected number of maxima lying above the value I_1 in an interval one second long.

We start on the first question by considering the random function

$$z = F(a_1, a_2, \cdots a_N; t)$$

where the a's are random variables. The probability that the random curve obtained by plotting z as a function of t increases through the value $z = z_1$ in the interval $t_1, t_1 + dt$ is[10]

$$dt \int_0^\infty \eta p(z_1, \eta; t_1)\, d\eta \tag{2.1}$$

where $p(\xi, \eta; t_1)$ denotes the probability density of the random variables

$$\xi = F(a_1, a_2, \cdots, a_N; t_1)$$

$$\eta = \left[\frac{\partial F}{\partial t}\right]_{t=t_1}.$$

In our case z becomes the current I defined by equation (1.1). The method used to obtain equation (3.3–9) of Reference A may also be used to show that the quantity $p(I_1, \eta, t_1)$ (which now appears in (2.1)) is given by

$$p(I_1, \eta, t_1) = \frac{\pi N_0}{-\psi_0''}\varphi(y - a\cos qt_1)\varphi(x + b\sin qt_1) \tag{2.2}$$

[10] This result is a straightforward generalization of expression (3.3-5) in Section 3.3 of Reference A where references to related results by M. Kac are given. A formula equivalent to (2.1) has also been given by Mr. H. Bondi in an unpublished paper written in 1944. He applies his formula to the problem studied in Section 4.

where $\varphi(\ )$ denotes the normal law function defined by equation (1.3) and

$$
\begin{aligned}
-\psi_0'' &= 4\pi^2 \int_0^\infty w(f) f^2\, df, \qquad N_0 = \frac{1}{\pi}\sqrt{\frac{-\psi_0''}{\psi_0}}, \qquad y = \frac{I_1}{\sqrt{\psi_0}}, \\
a &= \frac{Q}{\sqrt{\psi_0}}, \qquad x = \frac{\eta}{\sqrt{-\psi_0''}}, \qquad b = \frac{Qq}{\sqrt{-\psi_0''}} = \frac{2af_q}{N_0}.
\end{aligned} \tag{2.3}
$$

Equation (3.3–11) of Reference A shows that N_0 is the expected number of zeros per second which I_N would have if it were to flow alone.

Let $P(I_1, t_1)dt$ be the probability that I will increase through the value I_1 during the interval $t_1, t_1 + dt$. Then (2.1) and (2.2) give

$$
\begin{aligned}
P(I_1\ t_1) &= \int_0^\infty \eta p(I_1, \eta, t_1)\, d\eta \\
&= \pi N_0 \varphi(y - a\cos qt_1) \int_0^\infty x\varphi(x + b\sin qt_1)\, dx.
\end{aligned} \tag{2.4}
$$

The integral in (2.4) is of the form

$$
\begin{aligned}
\int_0^\infty x\varphi(x + v)\, dx &= \varphi(v) - v\int_v^\infty \varphi(x)\, dx \\
&= -\frac{v}{2} + \varphi(v) + v\int_0^v \varphi(x)\, dx \\
&= -v + \varphi(v) + v\varphi_{-1}(v) \\
&= -\frac{v}{2} + (2\pi)^{-1/2}\, {}_1F_1\left(-\frac{1}{2}; \frac{1}{2}; -\frac{v^2}{2}\right)
\end{aligned} \tag{2.5}
$$

where v replaces $b \sin qt_1$ and ${}_1F_1$ denotes a confluent hypergeometric function.

The distribution of the crossings at various portions of the cycle (of the sine wave) may be obtained by giving special values to qt_1 in (2.4).

The expected number of times I increases through the value I_1 in one second is

$$
\begin{aligned}
N(I_1) &= \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T P(I_1, t_1)\, dt_1 \\
&= N_0 \int_0^\pi \varphi(y - a\cos\theta)\left[\varphi(b\sin\theta) + b\sin\theta \int_0^{b\sin\theta} \varphi(x)\, dx\right] d\theta
\end{aligned} \tag{2.6}
$$

where we have used (2.4) and the second equation of (2.5). The integrand in (2.6) is composed of tabulated functions and is of a form suited to numerical integration. Expanding $\varphi(y - a\cos\theta)$ in (2.6) as in the derivation of (1.10), replacing the quantity within the brackets by the series shown in the last equation of (2.5) ,and integrating termwise leads to

$$N(I_1) = N_0(\pi/2)^{1/2} \sum_{n=0}^{\infty} \frac{\varphi^{(2n)}(y)}{n!n!} \left(\frac{a}{2}\right)^{2n} {}_1F_1\left(-\tfrac{1}{2}; n+1; -\frac{b^2}{2}\right) \tag{2.7}$$

The series (2.7) converges for all values of a, y, and b. This follows from the inequality (1.11) which may be applied to $\varphi^{(2n)}(y)$, and from the fact that the ${}_1F_1$ is less than $exp\ (b^2/2)$ as may be seen by comparing corresponding terms in their expansions.

The expected number of zeros, per second, of I is $2N(0)$ where we have set I_1, and hence y, equal to zero. In this case the integral in (2.6) may be simplified somewhat and we obtain

$$2N(0) = N_0\left[e^{-\alpha} I_0(\beta) + \frac{b^2}{2\alpha} Ie\left(\frac{\beta}{\alpha}, \alpha\right)\right] \tag{2.8}$$

where $I_0(\beta)$ is the Bessel function of order zero and imaginary argument and

$$\alpha = \frac{a^2 + b^2}{4}, \qquad \beta = \frac{a^2 - b^2}{4}.$$

$$Ie(k, x) = \int_0^x e^{-u} I_0(ku)\, du.$$

The integral $Ie(k, x)$ is tabulated in Appendix I.

I have been unable to obtain a simple derivation of (2.8). It was originally obtained from the following integral

$$N(I_1) = \frac{N_0}{2} \int_{-\pi}^{\pi} d\theta\, \varphi(y - a\cos\theta) \int_0^{\infty} x\varphi(x + b\sin\theta)\, dx \tag{2.9}$$

which may be derived from the second equation of (2.4) and the first of (2.6). Setting I_1 and y equal to zero and writing out the φ's gives

$$2N(0) = \frac{N_0}{2\pi} \int_{-\pi}^{\pi} d\theta \int_0^{\infty} dx$$
$$x \exp\left[-\tfrac{1}{2}(x^2 + 2bx\sin\theta + a^2\cos^2\theta + b^2\sin^2\theta)\right].$$

Equation (2.8) was obtained by applying the method of Appendix III to this expression.

3. Definitions and Simple Properties of R and Θ

The remaining portion of this paper is concerned with the envelope R and the corresponding phase angle θ. These quantities are introduced and some of their simpler properties discussed in this section.

Suppose that the frequency band associated with I_N is relatively narrow and contains the sine wave frequency f_q. The noise current may be resolved into two components, one "in phase" and the other "in quadrature" with $Q \cos qt$. Using the representation (2.8–6) of reference A and proceeding as in Section 3.7 of that paper:

$$I_N = \sum_{n=1}^{M} c_n \cos (\omega_n t - \varphi_n) \tag{3.1}$$

$$= \sum_{n=1}^{M} c_n \cos [(\omega_n - q)t - \varphi_n + qt]$$

$$= I_c \cos qt - I_s \sin qt \tag{3.2}$$

where

$$\begin{aligned} I_c &= \sum_{n=1}^{M} c_n \cos [(\omega_n - q)t - \varphi_n] \\ I_s &= \sum_{n=1}^{M} c_n \sin [(\omega_n - q)t - \varphi_n] \end{aligned} \tag{3.3}$$

$$\omega_n = 2\pi f_n, \qquad f_n = n\Delta f, \qquad c_n^2 = 2w(f_n)\Delta f$$

$w(f)$ denotes the power spectrum of I_N and the φ_n's are random variables distributed uniformly over the interval $(0, 2\pi)$.

The total current I may be written as

$$\begin{aligned} I &= Q \cos qt + I_N \\ &= (Q + I_c) \cos qt - I_s \sin qt \\ &= R \cos \theta \cos qt - R \sin \theta \sin qt \\ &= R \cos (qt + \theta) \end{aligned} \tag{3.4}$$

where we have introduced the envelope function R and the phase angle θ by means of

$$\begin{aligned} R \cos \theta &= Q + I_c \\ R \sin \theta &= I_s \end{aligned} \tag{3.5}$$

Since I_c and I_s are functions of t whose variations are relatively slow in comparison with those of $\cos qt$, the same is true of R and (usually) θ.

A graphical illustration of equations (3.4) and (3.5) which is often used is shown in Fig. 4.

In accordance with the usual convention used in alternating current theory, the vector OQ is supposed to be rotating about the origin O with angular velocity q. If I_N happened to have the frequency $q/2\pi$, its vector representation QT would be fixed relative to OQ. In general, however, the length and inclination of QT will change due to the random fluctuations of I_N. Thus the point T will wander around on the plane of the figure. If rms I_N is much less than Q, T will be close to the point Q most of the time. In this case

$$\begin{aligned} R &= [(Q + I_c)^2 + I_s^2]^{1/2} \sim Q + I_c \\ \theta &= \tan^{-1} \frac{I_s}{Q + I_c} \sim \frac{I_s}{Q} \\ \frac{d\theta}{dt} &\sim \frac{d}{dt}\frac{I_s}{Q} = \frac{I_s'}{Q} \end{aligned} \tag{3.6}$$

and a number of statistical properties of R and θ may be obtained from the corresponding properties of noise alone when we note that I_c, I_s, and I_s' behave like noise currents whose power spectra are concentrated in the lower portion of the frequency spectrum.

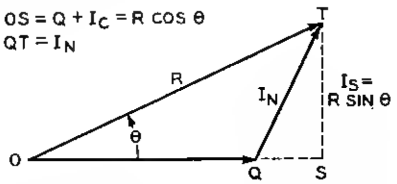

Fig. 4—Graphical representation of $I = Q\cos qt + I_N$.

By squaring both sides of equations (3.1) and (3.3) and then averaging with respect to t and the φ_n's we may show that I_c, I_s, and I_N all have the same rms value, namely $\psi_0^{1/2}$.

It may be seen from (3.3) that the power spectra of I_c and I_s are both given by

$$w(f_q + f) + w(f_q - f) \tag{3.7}$$

where it is assumed that $0 \leq f \ll f_q$. Likewise the power spectrum of the time derivative I_s' of I_s is

$$4\pi^2 f^2[w(f_q + f) + w(f_q - f)] \tag{3.8}$$

This follows from the representation of I_s' obtained by differentiating the expression (3.3) for I_s with respect to t, the procedure being the same as in the derivation of equation (7.2) in Section 7. The power spectra shown in Table 1 were computed from equations (3.7) and (3.8).

The correlation function for I_c, and hence also for I_s, is, from equations (A2–1) and (A2–3) of Appendix II,

$$\overline{I_c(t)I_c(t + \tau)} = g = \int_0^\infty w(f) \cos 2\pi(f - f_q)\tau \, df$$

where the bar denotes an average with respect to t and g is a function of τ. From (A2–3) the correlation function for I_s' is $-g''$ where the double prime on g denotes the second derivative with respect to τ.

Attention is sometimes fixed upon the variation in distance between successive zeros of I. The time between two successive zeros of I at, say, t_0 and t_1 is the time taken for $qt + \theta$, as appearing in $R \cos (qt + \theta)$, to increase by π. This assumes that the envelope R does not vanish in the interval. For the moment we write θ as $\theta(t)$ in order to indicate its dependence on the time t. Then t_0 and t_1 must satisfy the relation

$$qt_1 + \theta(t_1) - qt_0 - \theta(t_0) = \pi \tag{3.9}$$

Since $\theta(t)$ is a relatively slowly varying function we write

$$\theta(t_1) - \theta(t_0) = (t_1 - t_0)\theta'(t_0) + (t_1 - t_0)^2\theta''(t_0)/2 + \cdots$$

TABLE 1

POWER SPECTRA OF I_N, I_c, I_s, AND I_s'

I_N	I_c and I_s	I_s'
$w(f) = w_0 = \psi_0/\beta$ for $f_q - \beta/2 < f < f_q + \beta/2$ $w(f) = 0$ elsewhere f_q = mid-band frequency	$2w_0$ for $0 < f < \beta/2$ 0 elsewhere	$8\pi^2 f^2 w_0$ for $0 < f < \beta/2$ 0 elsewhere
$w(f) = w_0 = \psi_0/\beta$ for $f_q - \beta < f < f_q$ $w(f) = 0$ elsewhere f_q = top frequency	w_0 for $0 < f < \beta$ 0 elsewhere	$4\pi^2 f^2 w_0$ for $0 < f < \beta$ 0 elsewhere
$w(f) = \dfrac{\psi_0}{\sigma\sqrt{2\pi}} e^{-(f-f_q)^2/(2\sigma^2)}$	$\dfrac{2\psi_0}{\sigma\sqrt{2\pi}} e^{-f^2/(2\sigma^2)}$	$\dfrac{8\pi^2 f^2 \psi_0}{\sigma\sqrt{2\pi}} e^{-f^2/(2\sigma^2)}$

where the primes denote differentiation with respect to t. When this is placed in (3.9) and terms of order $(t_1 - t_0)^2$ neglected, we obtain

$$\frac{1}{2(t_1 - t_0)} = \frac{q}{2\pi} + \frac{1}{2\pi}\theta'(t_0) \tag{3.10}$$

which relates the interval between successive zeros to θ'.

The expression on the right hand side of (3.10) may be defined as the instantaneous frequency:

$$\text{Instantaneous frequency} = f_q + \frac{1}{2\pi}\frac{d\theta}{dt} \tag{3.11}$$

This definition is suggested when $\cos 2\pi ft$ is compared with $\cos (qt + \theta)$ and also by (3.10) when we note that the period of the instantaneous frequency is approximately equal to twice the distance between two successive zeros which is $2(t_1 - t_0)$.

The probability density of R is[11]

$$\frac{R}{\psi_0} \exp\left[-\frac{R^2 + Q^2}{2\psi_0}\right] I_0(RQ/\psi_0) \tag{3.12}$$

where $I_0\ (RQ/\psi_0)$ denotes the Bessel function of order zero with imaginary argument. In Section 3.10 of Reference A, it is shown that the average value of R^n is*

$$\overline{R^n} = (2\psi_0)^{n/2}\Gamma\left(\frac{n}{2} + 1\right) {}_1F_1\left(-\frac{n}{2}; \quad 1; \quad -\rho\right), \tag{3.13}$$

where $\rho = Q^2/(2\psi_0)$, of which special cases are

$$\begin{aligned} \bar{R} &= e^{-\rho/2}(\pi\psi_0/2)^{1/2}\ [(1 + \rho)I_0(\rho/2) + \rho I_1(\rho/2)] \\ \overline{R^2} &= Q^2 + 2\psi_0 \end{aligned} \tag{3.14}$$

Curves showing the distribution of R are also given there.

4. Expected Number of Crossings of R per Second

Here we shall obtain expressions for the expected number N_R of times, per second, the envelope passes through the value R with positive slope. When R is large, N_R is approximately equal to the expected number of maxima of the envelope per second exceeding R and when R is small N_R is approximately equal to the expected number of minima less than R. For the special case in which the noise band is symmetrical and is centered on the sine wave frequency f_q N_R is given by the relatively simple expression (4.8).

The probability that the envelope passes through the value R during the interval $t, t + dt$ with positive slope is, from (2.1),

$$dt \int_0^\infty R' p(R, R', t)\ dR' \tag{4.1}$$

where $p(R, R', t)$ denotes the probability density of R and its time derivative R', t being regarded as a parameter.

An expression for $p(R, R', t)$ may be obtained from the probability density of I_c, I_s, I_c', I_s'. From our representation of a noise current and the central limit theorem it may be shown (as is done for similar cases in Part III of Reference A) that the probability distribution of these four variables is

[11] In equation (60-A) of an unpublished appendix to his paper appearing in the *B.S.T.J.* Vol. 12 (1933), 35–75, Ray S. Hoyt gives an integral, obtained by integrating (3.12) with respect to R, for the cumulative distribution of R.

*The correlation function for the envelope of a signal plus noise, together with associated probability densities of the envelope and phase, is given by D. Middleton in a paper appearing soon in the Quart. Jl. of Appl. Math.

normal in four dimensions. If the variables be taken in the order given above the moment matrix is, from equations (A2–2) of Appendix II,

$$M = \begin{bmatrix} b_0 & 0 & 0 & b_1 \\ 0 & b_0 & -b_1 & 0 \\ 0 & -b_1 & b_2 & 0 \\ b_1 & 0 & 0 & b_2 \end{bmatrix} \tag{4.2}$$

where the b's are defined by the integrals in equations (A2–1). The inverse matrix is

$$M^{-1} = \frac{1}{B}\begin{bmatrix} b_2 & 0 & 0 & -b_1 \\ 0 & b_2 & b_1 & 0 \\ 0 & b_1 & b_0 & 0 \\ -b_1 & 0 & 0 & b_0 \end{bmatrix}, \quad B = b_0 b_2 - b_1^2 \tag{4.3}$$

which may be readily verified by matrix multiplication, and the determinant $| M |$ is B^2. The normal distribution may be written down at once when use is made of the formulas given in Section 2.9 of Reference A. The substitutions

$$\begin{aligned} I_c &= R\cos\theta - Q, & I_c' &= R'\cos\theta - R\sin\theta\,\theta' \\ I_s &= R\sin\theta, & I_s' &= R'\sin\theta + R\cos\theta\,\theta' \end{aligned} \tag{4.4}$$

$$dI_c dI_s dI_c' dI_s' = R^2 dR dR' d\theta d\theta'$$

enable us to write

$$\begin{aligned} b_2(I_c^2 + I_s^2) + b_0(I_c'^2 + I_s'^2) \\ -2b_1(I_c I_s' - I_s I_c') &= b_2(R^2 - 2QR\cos\theta + Q^2) \\ &\quad + b_0(R'^2 + R^2\theta'^2) \\ &\quad - 2b_1 R^2\theta' + 2b_1 Q(R'\sin\theta + R\theta'\cos\theta). \end{aligned}$$

Consequently the probability density of R, R', θ, θ' is

$$\begin{aligned} p(R, R', \theta, \theta') = \frac{R^2}{4\pi^2 B}\exp\Big\{-\frac{1}{2B}\,[b_2(R^2 - 2QR\cos\theta + Q^2) \\ + b_0(R'^2 + R^2\theta'^2) - 2b_1R^2\theta' + 2b_1Q(R'\sin\theta + R\theta'\cos\theta)]\Big\} \end{aligned} \tag{4.5}$$

In this expression R ranges from 0 to ∞, θ from $-\pi$ to π, and R' and θ' from $-\infty$ to $+\infty$. The probability density for R and R' is obtained by integrating (4.5) with respect to θ and θ' over their respective ranges. The integration with respect to θ' may be performed at once giving

$$p(R, R', t) = \frac{R(2\pi)^{-3/2}}{\sqrt{Bb_0}} \int_{-\pi}^{\pi} d\theta \exp\left\{-\frac{1}{2Bb_0}[B(R^2 - 2RQ\cos\theta + Q^2) + (b_0R' + b_1Q\sin\theta)^2]\right\} \tag{4.6}$$

From (4.1) and (4.6) it follows that the expected number N_R of times per second the envelope passes through R with positive slope is

$$N_R = \frac{R(2\pi)^{-3/2}}{\sqrt{Bb_0}} \int_{-\pi}^{\pi} d\theta \int_0^{\infty} R'dR' \exp\left\{-\frac{1}{2Bb_0}[B(R^2 - 2RQ\cos\theta + Q^2) + (b_0R' + b_1Q\sin\theta)^2]\right\} \tag{4.7}$$

When the power spectrum $w(f)$ of the noise current I_N is symmetrical about the sine wave frequency f_q, b_1 is zero and B is equal to b_0b_2. In this case the integrations in (4.7) may be performed. We obtain

$$\begin{aligned} N_R &= \left(\frac{b_2}{2\pi}\right)^{1/2} \frac{R}{b_0} I_0\left(\frac{RQ}{b_0}\right) \exp\left(-\frac{R^2 + Q^2}{2b_0}\right) \\ &= \left(\frac{b_2}{2\pi}\right)^{1/2} \times \left[\begin{matrix}\text{Probability density of} \\ \text{envelope at the value } R\end{matrix}\right] \end{aligned} \tag{4.8}$$

where the second line is obtained from expression (3.12). As will be seen from its definition (A2–1), b_0 is equal to the mean square value ψ_0 of I_N (and also of I_c and I_s).

Introducing the notation

$$\begin{aligned} v &= Rb_0^{-1/2} = R/\text{rms } I_N \\ a &= Ab_0^{-1/2} = Q/\text{rms } I_N, \end{aligned} \tag{4.9}$$

which is the same as that of equations (3.10–15) of Reference A except that there P denotes the amplitude of the sine wave and plays the same role as Q does here, enables us to write (4.8) as

$$N_R = \left[\frac{b_2}{2\pi b_0}\right]^{1/2} vI_0(av)e^{-(v^2+a^2)/2} = \left[\frac{b_2}{2\pi b_0}\right]^{1/2} p(v). \tag{4.10}$$

The function $p(v)$ is plotted as a function of v for various values of a in Fig. 6, Section 3.10, of Reference A.

It is interesting to note that

$$(b_2/b_0)^{1/2}/\pi = \text{Expected number of zeros per second of } I_c \text{ (or of } I_s) \tag{4.11}$$

This relation, which is true even if the noise band is not symmetrical about f_q, follows from equation (3.3–11) of Reference A.

When $Q \gg \text{rms } I_N$ and f_q is not at the center of the noise band it is easier to obtain the asymptotic form of N_R from the approximation (3.6),

$$R \sim Q + I_c,$$

instead of the double integral (4.7). When $Q >> \text{rms } I_N$ and R is in the neighborhood of Q (as it is most of the time in this case), N_R is approximately equal to the expected number of times I_c increases through tbe value $I_{c1} = R - Q$ in one second. Thus, regarding I_c as a random noise current we have from expression (3.3–14) of Reference A

$N_R \sim e^{-I_{c1}^2/(2b_0)} \times$ [1/2 the expected number of zeros per second of I_c]

and when we use equation (4.11) we obtain

$$N_R \sim \frac{1}{2\pi}(b_2/b_0)^{1/2} e^{-(R-Q)^2/(2b_0)} = \frac{1}{2\pi}(b_2/b_0)^{1/2} e^{-(v-a)^2/2} \tag{4.12}$$

Table 2

$w(f) = w_0 = b_0/\beta$ over a Band of Width β

	b_2	N_R
1. Band extends from $f_q - \beta/2$ to $f_q + \beta/2$	$\pi^2\beta^2 b_0/3$	$(\pi/6)^{1/2}\beta p(v) = 0.724\,\beta p(v)$
2. Same as 1 and in addition $Q = 0$	"	$(\pi/6)^{1/2}\beta v e^{-v^2/2}$
3. Same as 1 and in addition $Q >> \text{rms } I_N$	"	$\sim \frac{\beta}{2\sqrt{3}} e^{-(v-a)^2/2}$
4. Band extends from f_q to $f_q + \beta$ and $Q >> \text{rms } I_N$	$4\pi^2\beta^2 b_0/3$	$\sim \frac{\beta}{\sqrt{3}} e^{-(v-a)^2/2}$

Table 2 lists the forms assumed by (4.10) and (4.12) when the power spectrum $w(f)$ of the noise current I_N is constant over a frequency band of width β. The quantity b_0 in the expressions for b_2 represents the mean square value of I_N.

In the general case where the band of noise is not centered on f_q and where R is not large enough to make (4.12) valid we are obliged to return to the double integral (4.7). Some simplification is possible, but not as much as could be desired. Introducing the notation

$$\alpha = RQ/b_0, \qquad \gamma = b_1 Q(Bb_0)^{-1/2}$$

$$x = (b_0 R' + b_1 Q \sin\theta)(Bb_0)^{-1/2}$$

enables us to write (4.7) as

$$N_R = (2\pi)^{-3/2}(R/b_0)(B/b_0)^{1/2}e^{-(R^2+Q^2)/(2b_0)}$$
$$\int_{-\pi}^{\pi} d\theta \int_{\gamma \sin\theta}^{\infty} (x - \gamma \sin\theta)e^{\alpha\cos\theta - x^2/2}\, dx \tag{4.13}$$

Part of the integrand may be integrated with respect to x and the remaining portion integrated by parts with respect to θ. The double integral in the second line of (4.13) then becomes

$$\int_{-\pi}^{\pi} e^{\alpha\cos\theta-(\gamma\sin\theta)^2/2}\, d\theta + \int_{-\pi}^{\pi}\left[\int_{\gamma\sin\theta}^{\infty} e^{-x^2/2}\, dx\right] d[\gamma\alpha^{-1}e^{\alpha\cos\theta}]$$
$$= \int_{-\pi}^{\pi} (1 + \gamma^2\alpha^{-1}\cos\theta)e^{\alpha\cos\theta-(\gamma\sin\theta)^2/2}\, d\theta$$
$$= \gamma\alpha^{-1}e^{-(\gamma^2+\alpha^2\gamma^{-2})/2}\int_{-\pi}^{\pi}(\gamma\cos\theta + \alpha/\gamma)e^{(\gamma\cos\theta+\alpha/\gamma)^2/2}\, d\theta \tag{4.14}$$
$$= 2\pi\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!}\left(-\frac{\gamma^2}{\alpha}\right)^n [I_n(\alpha) + \gamma^2\alpha^{-1}I_{n+1}(\alpha)].$$

The series is obtained by expanding exp $[-(\gamma \sin\theta)^2/2]$ in the second equation in powers of $\sin\theta$ and integrating termwise.

5. Probability Density of $\frac{d\theta}{dt}$

As was pointed out in Section 3 the time derivative θ' of the phase angle θ associated with the envelope is closely related to the instantaneous frequency. The probability density $p(\theta')$ of θ' may be expressed in terms of modified Bessel functions as shown by equation (5.4). Curves for $p(\theta')$ are given when the sine wave frequency f_q lies at the middle of a symmetrical band of noise. Although the expressions for $p(\theta')$ are rather complicated, those for the averages $\bar{\theta}'$ and $\overline{|\theta'|}$ given by equations (5.7) and (5.16) are relatively simple.

The probability density $p(\theta')$ may be obtained by integrating the expression (4.5) for $p(R, R', \theta, \theta')$ with respect to R, R', θ. The integration with respect to R', the limits being $-\infty$ and $+\infty$, gives the probability density for R, θ, θ':

$$p(R, \theta, \theta') = \frac{R^2}{4\pi^2}\left(\frac{2\pi}{b_0 B}\right)^{1/2} \exp\,[-aR^2 + 2bR\cos\theta + c\sin^2\theta - b_2Q^2/(2B)] \tag{5.1}$$

where

$$
\begin{aligned}
B &= b_0 b_2 - b_1^2 & b &= Q(b_2 - b_1\theta')/(2B) \\
a &= (b_2 - 2b_1\theta' + b_0\theta'^2)/(2B) & c &= Q^2 b_1^2/(2Bb_0) = b_1^2\rho/B \\
\rho &= Q^2/(2b_0) & \gamma &= b^2/a
\end{aligned}
\tag{5.2}
$$

and b_0, b_1, b_2 are given in Appendix II.

Integrating with respect to R gives the probability density for θ, θ'. Expanding $\exp(2bR\cos\theta)$ in powers of R and integrating termwise,

$$
p(\theta, \theta') = \frac{1}{16\pi a}\left(\frac{2}{ab_0B}\right)^{1/2} e^{c\sin^2\theta - b_2b_0\rho/B} \sum_{n=0}^{\infty} \frac{n+1}{\Gamma\left(\frac{n}{2}+1\right)}\left(\frac{b\cos\theta}{a^{1/2}}\right)^n \tag{5.3}
$$

When we integrate θ from $-\pi$ to π to obtain $p(\theta')$ the terms for which n is odd disappear and we have to deal with the series, writing γ for b^2/a,

$$
\sum_{m=0}^{\infty} \frac{2m+1}{m!}(\gamma\cos^2\theta)^m = (2\gamma\cos^2\theta + 1)\exp(\gamma\cos^2\theta)
$$

Thus, the probability density of θ' is

$$
\begin{aligned}
p(\theta') &= \frac{1}{16\pi a}\left(\frac{2}{ab_0B}\right)^{1/2}\int_{-\pi}^{\pi}(2\gamma\cos^2\theta + 1)\, e^{c\sin^2\theta + \gamma\cos^2\theta - b_2b_0\rho/B}\, d\theta \\
&= \frac{1}{8a}\left(\frac{2}{ab_0B}\right)^{1/2}\left[(\gamma+1)I_0\left(\frac{\gamma-c}{2}\right) + \gamma I_1\left(\frac{\gamma-c}{2}\right)\right] \exp\left[\frac{c+\gamma}{2} - \frac{b_2b_0\rho}{B}\right]
\end{aligned}
\tag{5.4}
$$

From (5.2)

$$
\begin{aligned}
\gamma - c &= \rho\,\frac{b_2 - 2b_1\theta'}{b_2 - 2b_1\theta' + b_0\theta'^2} \\
\frac{c+\gamma}{2} - \frac{b_2b_0\rho}{B} &= -\frac{\rho}{2}\,\frac{b_2 - 2b_1\theta' + 2b_0\theta'^2}{b_2 - 2b_1\theta' + b_0\theta'^2}
\end{aligned}
\tag{5.5}
$$

It will be noted that for large values of $|\theta'|$ the probability density of θ' varies as $|\theta'|^{-3}$. Although this makes the mean square value of θ' infinite, the average values $\overline{\theta'}$ and $\overline{|\theta'|}$ of θ' and $|\theta'|$ still exist. In order to obtain $\overline{\theta'}$ it is convenient to return to (4.5) and write

$$
\overline{\theta'} = \int_{-\pi}^{\pi} d\theta \int_0^{\infty} dR \int_{-\infty}^{+\infty} dR' \int_{-\infty}^{\infty} d\theta'\, \theta' p(R, R', \theta, \theta') \tag{5.6}
$$

The integration with respect to θ' may be performed by setting $R\theta'$ equal to x and using

$$\int_{-\infty}^{+\infty} x\, e^{-\alpha x^2 + 2\beta x}\, dx = (\beta/\alpha)(\pi/\alpha)^{1/2}\, e^{\beta^2/\alpha}$$

The integral in R' reduces to a similar integral except that the factor x in the integrand is absent. Performing these two integrations and using the definition of B leads to

$$\overline{\theta'} = \frac{1}{2\pi}\frac{b_1}{b_0^2}\int_{-\pi}^{\pi} d\theta \int_0^{\infty} dR\,(R - Q\cos\theta) \cdot \exp\left[-\frac{1}{2b_0}(R^2 - 2QR\cos\theta + Q^2)\right]$$

We may integrate at once with respect to R. When this is done $\cos\theta$ disappears and the integration witb respect to θ becomes easy. Thus

$$\overline{\theta'} = (b_1/b_0)\exp[-Q^2/(2b_0)] = (b_1/b_0)e^{-\rho} \tag{5.7}$$

When the noise power spectrum is equal to w_0 in the band extending from $f_0 - \beta/2$ to $f_0 + \beta/2$ and is zero outside the band, $b_1 = 2\pi(f_0 - f_q)b_0$. Hence, from (3.11),

$$\begin{aligned}\text{ave. instantaneous frequency} &= f_q + \overline{\theta'}/(2\pi) \\ &= f_0 + (f_q - f_0)(1 - e^{-\rho})\end{aligned} \tag{5.8}$$

In the remainder of this section we assume tbe power spectrum of the noise current to be symmetrical about the sine wave frequency f_q. In this case b_1 and c are zero, B is equal to $b_0 b_2$ and (5.4) becomes

$$\begin{aligned}p(\theta') &= \tfrac{1}{2}(b_0/b_2)^{1/2}(1 + z^2)^{-3/2}e^{-\rho+y/2} \\ &\quad [(y+1)I_0(y/2) + yI_1(y/2)] \\ &= \tfrac{1}{2}(b_0/b_2)^{1/2}(1 + z^2)^{-3/2}e^{-\rho}\,{}_1F_1\left(\frac{3}{2}; 1; y\right)\end{aligned} \tag{5.9}$$

where ${}_1F_1$ denotes a confluent hypergeometric function[12] and

$$z^2 = b_0\theta'^2/b_2, \qquad y = (\gamma)_{b_1=0} = \rho/(1 + z^2) \tag{5.10}$$

When the noise power spectrum is constant in the band extending from $f_q - \beta/2$ to $f_q + \beta/2$ (see Table 2, Section 4)

$$(b_2/b_0)^{1/2} = 3^{-1/2}\beta\pi, \qquad z = 3^{1/2}\theta'/(\beta\pi) \tag{5.11}$$

[12] The relation used above follows from equation (66) (with misprint corrected) of W. R. Bennett's paper cited in connection with equation (1.2).

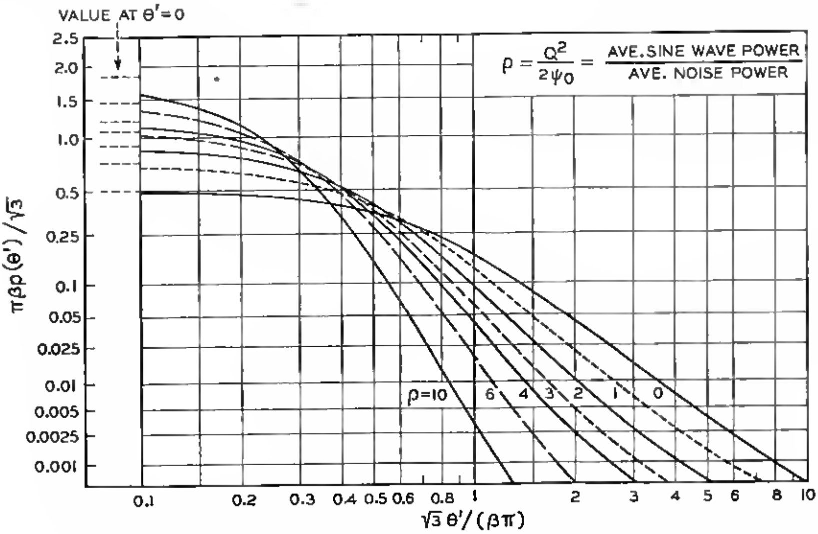

Fig. 5—Probability density of time derivative of phase angle.

$p(\theta')\, d\theta'$ = probability that the value of $d\theta/dt$ at an instant selected at random lies between θ' and $\theta' + d\theta'$. The power spectrum of I_N is constant in band of width β centered on f_q and is zero outside this band.

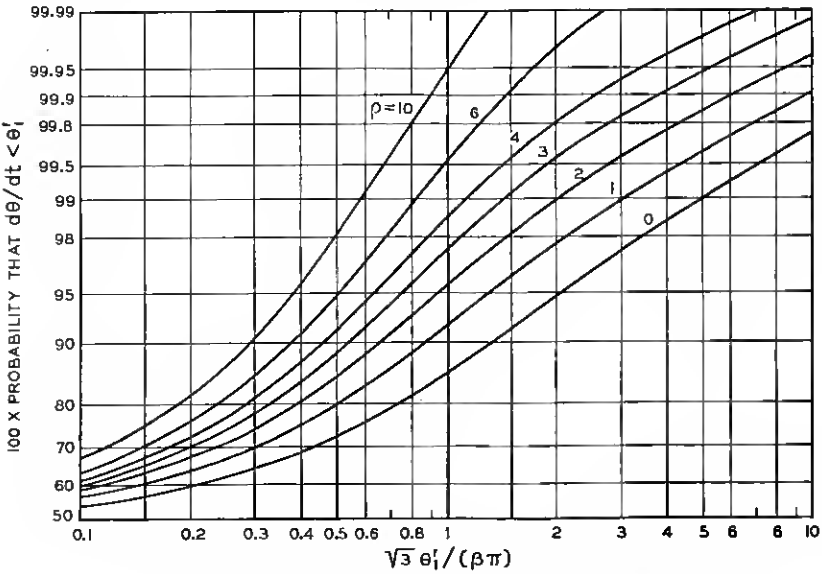

Fig. 6—Cumulative distribution of time derivative of phase angle.

Notation explained in Fig. 5.

The probability density $p(\theta')$ of θ' and its cumulative distribution, obtained by numberical integration, are shown in Figs. 5 and 6.

The probability that θ' exceeds a given θ_1' is equal to the probability that z exceeds z_1, where z_1 denotes $(b_0/b_2)^{1/2}\theta_1'$, and both probabilities are equal to

$$\frac{e^{-\rho}}{2}\int_{z_1}^{\infty}(1+z^2)^{-3/2}\,{}_1F_1\left[\frac{3}{2};1;\rho(1+z^2)^{-1}\right]dz \tag{5.12}$$

The probability that $\theta' > \theta_1'$ becomes $e^{-\rho}/(4z_1^2)$ as $\theta_1' \to \infty$.

When $Q \gg$ rms I_N the leading term in the asymptotic expansion of the ${}_1F_1$ in (5.9) gives

$$p(\theta') \sim \frac{1}{\sigma\sqrt{2\pi}}e^{-\theta'^2/(2\sigma^2)}, \qquad \sigma^2 = b_2/Q^2 \tag{5.13}$$

when it is assumed that $z^2 \ll 1$. This expression holds only for the central portion of the curve for $p(\theta')$. Far out on the curve, $p(\theta')$ still varies as θ'^{-3}. Equation (5.13) may be obtained directly by using the approximation (3.6) that θ' is nearly equal to I_s'/Q and noticing that b_2 is the mean square value of I_s'.

If the sine wave is absent, ρ is zero and

$$p(\theta') = \tfrac{1}{2}(b_0/b_2)^{1/2}(1+z^2)^{-3/2} \tag{5.14}$$

which is consistent with the results given between equations (3.4–10) and (3.4–11) of Reference A. In this case (5.12) becomes

$$\frac{1}{2} - \frac{z_1}{2}(1+z_1^2)^{-1/2} \tag{5.15}$$

Although the standard deviation of θ' is infinite an idea of the spread of the distribution may be obtained from the average value of $|\theta'|$. Setting b_1 equal to zero in (4.5) in order to obtain the case in which the noise band is symmetrical about the sine-wave frequency leads to

$$\overline{|\theta'|} = \frac{2}{4\pi^2 b_0 b_2}\int_0^{\infty} dR \int_{-\pi}^{\pi} d\theta \int_{-\infty}^{+\infty} dR' \int_0^{\infty} d\theta'\,\theta'\,R^2$$
$$\exp \tfrac{1}{2}\left[-(R^2 - 2QR\cos\theta + Q^2)/b_0 - (R'^2 + R^2\theta'^2)/b_2\right]$$

The integrals in R', θ' cause no difficulty and the integral in θ is proportional to the Bessel function $I_0(QR/b_0)$. When the resulting integral in R is evaluated[13] we obtain

$$\overline{|\theta'|} = (b_2/b_0)^{1/2}e^{-\rho/2}I_0(\rho/2) \tag{5.16}$$

where $\rho = Q^2/(2b_0)$.

[13] See, for example, G. N. Watson, "Theory of Bessel Functions," Cambridge (1944), p. 394, equation (5).

When ρ is zero equation (5.16) agrees with a result given in Section 3.4 of reference A, namely, for an ideal band pass filter

$$\frac{\text{ave} \, |\tau - \tau_1|}{\tau_1} = \frac{f_b - f_a}{\sqrt{3}(f_b + f_a)}$$

where τ is the interval between two successive zeros and τ_1 is its average value. τ is equal to $t_1 - t_0$ of our equation (3.10) from which it follows that

$$(\tau - \tau_1)/\tau_1 \approx -\theta'/q \tag{5.17}$$

6. Expected Number of Crossings of θ and $\frac{d\theta}{dt}$ per Second

After a brief study of the expected number of times per second the phase angle θ increases through 0 and through π (where it is assumed that $-\pi < \theta \leq \pi$) expressions are obtained for the expected number $N_{\theta'}$ of times per second the time derivative of θ increases through the value θ'.

The point T shown in Fig. 4 of Section 3 wanders around, as time goes by, in the plane of the figure. How many times may we expect it to cross some preassigned section of the line OQ in one second? To answer this problem we note that, from expression (2.1), the probability that θ increases through zero during the interval t, $t + dt$ with the envelope lying between R and $R + dR$ is

$$dt \, dR \int_0^\infty \theta' p(R, 0, \theta') \, d\theta' \tag{6.1}$$

where the probability density in the integrand is obtained by setting θ equal to zero in equation (5.1). The expected number of such crossings per second is

$$\begin{aligned}(2\pi)^{-3/2} (b_0 B)^{-1/2} R^2 dR e^{-b_2(R-Q)^2/(2B)} \\ \int_0^\infty d\theta' \, \theta' \exp \left[-b_0 R^2 \theta'^2/(2B) + b_1 R(R - Q)\theta'/B\right]\end{aligned} \tag{6.2}$$

which may be evaluated in terms of error functions or the function $\varphi_{-1}(x)$ defined by equation (1.8). For the special case in which the power spectrum of the noise current I_N is symmetrical about the sine wave frequency, b_1 is zero and (6.2) yields

$$(2\pi)^{-3/2} b_0^{-1} b_2^{1/2} e^{-(R-Q)^2/(2b_0)} \, dR \tag{6.3}$$

From equation (6.1) onwards we have tacitly assumed that the range of θ is given by $-\pi < \theta \leq \pi$ because setting θ equal to any multiple of 2π in our equations leads to the same result as setting θ equal to zero. This is due to θ occurring only in $\cos \theta$ and $\sin \theta$. When θ increases through the value π, as it does when the point T crosses, in the downward direction, the extension of the line OQ lying to the left of the point O in Fig. 4, we imagine the value of θ to change discontinuously to the value $-\pi$.

The expected number of times per second θ increases through π may be obtained from (6.2) and, in the symmetrical case, from (6.3) by changing the sign of Q since this produces the same effect as changing θ from 0 to π in $p(R, \theta, \theta')$.

The expected number of crossings per second when R lies between two assigned values may be obtained by integrating the above equations. For example, the number of times per second θ increases through zero with R between Q and R_1 is, from (6.3) for the symmetrical case,

$$(4\pi)^{-1}(b_2/b_0)^{1/2} \operatorname{erf} [(2b_0)^{-1/2} \mid R_1 - Q \mid] \tag{6.4}$$

where we have used the absolute value sign to indicate that R_1 may be either less than or greater than Q and

$$\operatorname{erf} x = 2\pi^{-1/2} \int_0^x e^{-t^2}\, dt \tag{6.5}$$

Expressions for b_0 and b_2 are given by equations (A2–1) of Appendix II. The mean square value of I_N is b_0, and when the power spectrum of I_N is constant over a band of width β, $b_2 = \pi^2\beta^2 b_0/3$.

In much the same way it may be shown that the expected number of times per second θ increases through π with R between 0 and R_1 is

$$(4\pi)^{-1}(b_2/b_0)^{1/2} \{\operatorname{erf} [(2b_0)^{-1/2}(R_1 + Q)] - \operatorname{erf} [(2b_0)^{-1/2}Q]\} \tag{6.6}$$

A check on these equations may be obtained by noting that the expected number of zeros per second of I_s, given by equation (4.11), is equal to twice the number of times θ increases through zero plus twice the number of times θ increases through π. Setting R_1 equal to zero in (6.4), to infinity in both (6.4) and (6.6), and adding the three quantities obtained gives half of (4.11), as it should.

Now we shall consider the crossings of θ'. The equations in the first part of the analysis are quite similar to those encountered in Section 3.8 of Reference A where the maxima of R, for noise alone, are discussed. We start by introducing the variables $x_1, x_2, \cdots x_6$ where

$$x_1 = I_c = R\cos\theta - Q, \qquad x_4 = I_s = R\sin\theta \tag{6.7}$$

and the remaining x's are defined in terms of the derivatives of I_c and I_s and are given by the equations just below (3.8–4) of Reference A.

Here we shall consider the noise band to be symmetrical about the sine wave frequency f_q so that b_1 and b_3 are zero. Then from equations (3.8–3) and (3.8–4) of Reference A the probability density of $x_1, x_2, \cdots x_6$ is

$$\frac{1}{8\pi^3 b_2 D} \exp\left(-\frac{1}{2D}[b_4(x_1^2 + x_4^2) + 2b_2(x_1 x_3 + x_4 x_6) + (D/b_2)(x_2^2 + x_5^2) + b_0(x_3^2 + x_6^2)]\right) \tag{6.8}$$

where $D = b_0 b_4 - b_2^2$ and the b_n's are given by equations (A2–1). Replacing the x's by their expressions in terms of R and θ, similar to those just above equation (3.8–5) of Reference A, shows that the probability density for $R, R', R'', \theta, \theta', \theta''$ is

$$\begin{aligned} p(R, R', R'', \theta, \theta', \theta'') = \frac{R^3}{8\pi^3 b_2 D} \exp\Big(-\frac{1}{2D}[b_4(R^2 - 2RQ\cos\theta + Q^2) \\ + (D/b_2)(R'^2 + R^2\theta'^2) + 2b_2(RR'' - R^2\theta'^2) \\ + b_0(R''^2 - 2RR''\theta'^2 + 4R'^2\theta'^2 + 4RR'\theta'\theta'' + R^2\theta'^4 + R^2\theta''^2) \\ - 2b_2 Q(R''\cos\theta - R\theta'^2\cos\theta - 2R'\theta'\sin\theta - R\theta''\sin\theta)]\Big) \end{aligned} \tag{6.9}$$

It must be remembered that (6.9) applies only to the symmetrical case in which b_1 and b_3 are zero.

Integrating R' and R'' in (6.9) from $-\infty$ to ∞ gives the probability density of $R, \theta, \theta', \theta''$. The integration with respect to R'' is simplified by changing to the variable $R'' - R\theta'^2$. The result is

$$\begin{aligned} p(R, \theta, \theta', \theta'') = R^3(2\pi)^{-2}(b_0 b_2 D)^{-1/2}(1 + u)^{-1/2} \\ \exp\left(-\frac{1}{2b_0}\left[R^2 - 2RQ\cos\theta + Q^2 + b_0 R^2\theta'^2/b_2 + \frac{(Qb_2\sin\theta + b_0 R\theta'')^2}{(1 + u)D}\right]\right) \end{aligned} \tag{6.10}$$

where $u = 4b_2 b_0 \theta'^2/D$. The expected number of times per second the time derivative of θ increases through the value θ' is

$$\begin{aligned} N_{\theta'} &= \int_{-\pi}^{\pi} d\theta \int_0^\infty dR \int_0^\infty d\theta''\, \theta'' p(R, \theta, \theta', \theta'') \\ &= \pi^{-2}(b_2\delta/b_0)^{1/2} \int_{-\pi}^{\pi} d\theta \int_0^\infty r\,dr \int_0^\infty x\,dx \\ &\quad \exp[-\gamma r^2 + 2r\alpha\cos\theta - \alpha^2 - \delta(x + \alpha\sin\theta)^2] \end{aligned} \tag{6.11}$$

where we have set

$$
\begin{aligned}
r &= R(2b_0)^{-1/2} & x &= rb_0\theta''/b_2 \\
\alpha &= Q(2b_0)^{-1/2} = \rho^{1/2} & \gamma &= 1 + b_0\theta'^2/b_2 = 1 + z^2 \qquad (6.12)\\
& & \delta &= \frac{b_2^2}{(1+u)D}
\end{aligned}
$$

r being regarded as a constant when the variable of integration is changed from θ'' to x.

The double integral in θ and x occurring in (6.11) is of the same form as (A3–1) of Appendix III and hence may be transformed into (A3–3). Here $a = r\,\alpha$, $c = -\delta\alpha^2$, $c + b^2 = 0$. The diameter of the path of integration C may he chosen so large that the order of integration may be interchanged and the integration with respect to r performed. The result is again an integral of the form (A3–3) in which $a^2 = 0$. When this is reduced to (A3–6) it becomes

$$
\begin{aligned}
N_{\theta'} = {} & e^{-\rho}(2\pi\gamma)^{-1}b_2^{1/2}(b_0\delta)^{-1/2}\,[e^{-\delta\rho/2}I_0(\delta\rho/2) \\
& + (1+\gamma\delta)(1+\gamma\delta/2)^{-1}e^{\rho/\gamma}Ie\,\{\gamma\delta(2+\gamma\delta)^{-1},\ \rho/\gamma + \delta\rho/2\}]
\end{aligned} \qquad (6.13)
$$

where we have used $Ie(-k, x) = Ie(k, x)$ which follows from the definition (A1–1) given in Appendix I.

When there is no sine wave present, ρ is zero and (6.13) becomes

$$
N_{\theta'} = \frac{1}{2\pi\gamma}\left(\frac{b_2}{b_0\delta}\right)^{1/2} = \frac{\sqrt{\dfrac{b_4}{b_2} - \dfrac{b_2}{b_0} + 4\theta'^2}}{2\pi\left(1 + \dfrac{b_0}{b_2}\theta'^2\right)} \qquad (6.14)
$$

This gives a partial check on some of the above analysis since (6.14) may be obtained immediately by setting α equal to zero in (6.11). Another check may be ohtained by letting $\rho \to \infty$ and using $Ie(k, \infty) = (1 - k^2)^{-1/2}$. (6.13) becomes

$$
N_{\theta'} \sim (2\pi)^{-1}(b_4/b_2)^{1/2}e^{-\rho z^2} \qquad (6.15)
$$

which agrees with the result ohtained from $\theta' \sim I_s'/Q$.

For the case in which the power spectrum $w(f)$ of the noise is equal to the constant value w_0 over the frequency band extending from $f_q - \beta/2$ to $f_q + \beta/2$,

$$
b_0 = \beta w_0, \quad b_2 = \pi^2\beta^3 w_0/3 = \pi^2\beta^2 b_0/3, \quad b_4 = \pi^4\beta^5 w_0/5 = \pi^4\beta^4 b_0/5 \qquad (6.16)
$$

These lead to

$$z = (b_0/b_2)^{1/2}\theta' = 3^{1/2}\theta'/(\pi\beta) \qquad D/b_2^2 = b_4b_0/b_2^2 - 1 = 9/5 - 1 = 4/5$$

$$u = 4\, b_2^2 z^2/D = 5z^2 \qquad \delta = \frac{5}{4(1 + 5z^2)} \tag{6.17}$$

$$\gamma = 1 + z^2$$

and the coefficient in (6.13) may be simplified by means of

$$\frac{1}{2\pi\gamma}\left(\frac{b_2}{b_0\delta}\right)^{1/2} = \frac{\beta}{1+z^2}\left(\frac{1+5z^2}{15}\right)^{1/2} \tag{6.18}$$

From (6.14) we see that (6.18) is equal to $N_{\theta'}$ when noise alone is present (and is of constant strength in the band of width β). The curves of $N_{\theta'}/\beta$ versus z shown in Fig. 7 were obtained by setting (6.17) and (6.18) in (6.13). $N_{\theta'}/\beta$ approaches $e^{-\rho}/(z\sqrt{3})$ as $z \to \infty$.

When the wandering point T of Fig. 4 passes close to the point O, θ changes rapidly by approximately π and produces a pulse in θ'. In discussions of frequency modulation θ' is sometimes regarded as a noise voltage which is applied to a low pass filter. Although the closer T comes to O the higher the pulse, the area under the pulse will be of the order of π and the response of the low pass filter may be calculated approximately.

That the pulses in θ' arise in the manner assumed above may be checked as follows. We choose a point relatively far out on the curve for $\rho = 5$ in Fig. 7, say $z = \sqrt{3}\theta'/(\beta\pi) = 1.6$ or $\theta' = 2.9\beta$. The number of pulses per second having peaks higher than 2.9β is roughly $N_{\theta'} = .009\beta$, and half of these have peaks greater than $\theta' = 3.8\beta$ which is obtained from Fig. 7 for $N_{\theta'} = .0045\beta$. From Fig. 6 we see that θ' exceeds 2.9β about .0018 of the time. Thus the average width at the height 2.9β of the class of pulses whose peaks exceed this value is $.0018/(.009\beta) = .2/\beta$ seconds. This figure is to be checked by the width obtained from the assumption that the typical pulse arises when T moves along a straight line with speed v and passes within a distance b of O. We take $\tan\theta = vt/b = \alpha t$ so that $\theta' = \alpha/(1 + \alpha^2t^2)$. From this expression for θ' it follows that a pulse of peak height 3.8β (the median height) has a width of $.3/\beta$ seconds at $\theta' = 2.9\beta$. This agreement seems to be fairly good in view of the roughness of our work. A similar comparison may be made for $\rho = 0$ by using the limiting forms of (5.15) and (6.18). Here it is possible to compute the average width instead of estimating it from the median peak value. Exact agreement is obtained, both methods leading to an average width of $\pi/(4\theta')$ seconds at height θ'.

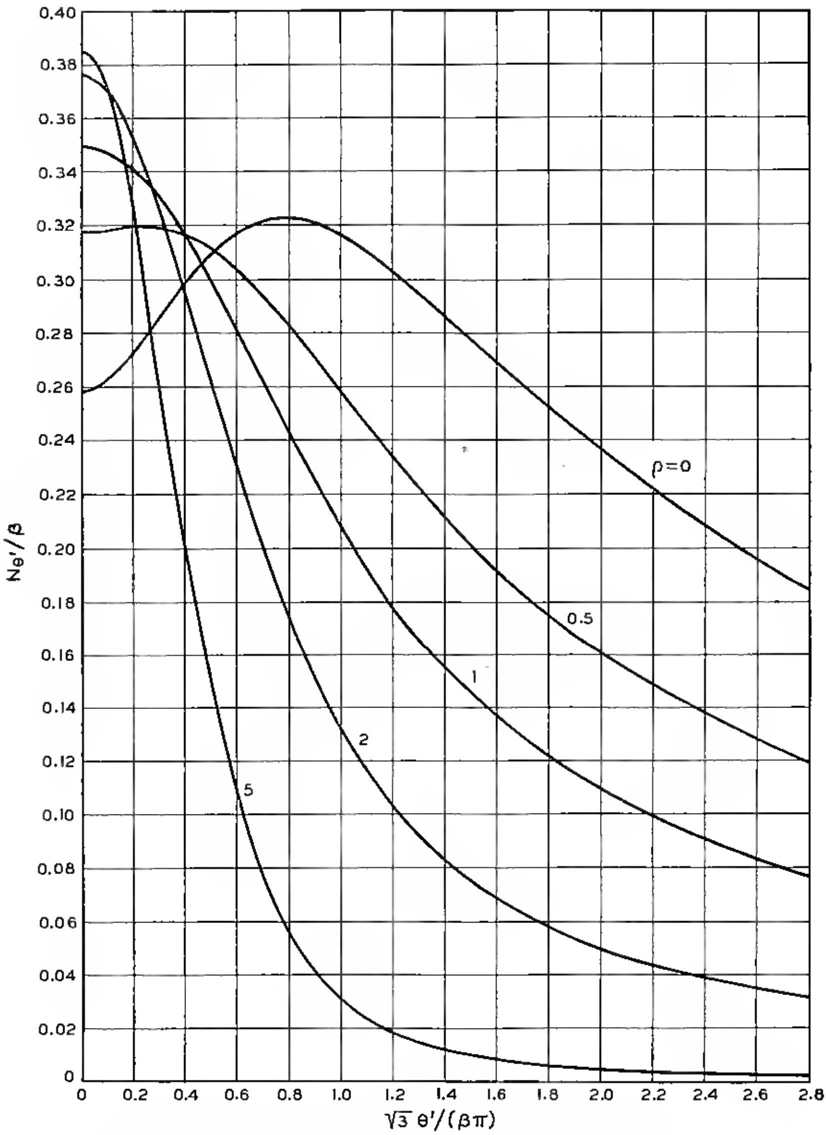

Fig. 7—Crossings of time derivative of phase angle.

$N_{\theta'}$ = expected number of times per second $d\theta/dt$ increases through the value θ'. ρ, β, and the power spectrum of I_N are the same as in Fig. 5.

7. Correlation Function for $\dfrac{d\theta}{dt}$

In this section we shall compute the correlation function $\Omega(\tau)$ of $\theta'(t)$. We are primarily interested in $\Omega(\tau)$ because it is, according to a fundamental result due to Wiener, the Fourier transform[14] of the power spectrum $W(f)$ of $\theta'(t)$.

We shall first consider the case in which the sine wave power is very large compared with the noise power so that, from (3.6), θ is approximately I_s/Q and θ' approximately I_s'/Q. Then using (A2–3) and (A2–1)

$$\begin{aligned}\Omega(\tau) &= \overline{\theta'(t)\theta'(t+\tau)} \approx Q^{-2}\overline{I_s'(t)I_s'(t+\tau)} \\ &= -g''Q^{-2} = 4\pi^2Q^{-2}\int_0^\infty w(f)(f-f_q)^2 \cos 2\pi(f-f_q)\tau\, df\end{aligned} \tag{7.1}$$

When $w(f)$ is effectively zero outside a relatively narrow band in the neighborhood of f_q, as it is in the cases with which we shall deal, (7.1) leads to the relation (divide the interval $(0, \infty)$ into $(0, f_q)$ and (f_q, ∞), introduce new variables of integration $f_1 = f_q - f$, $f_2 = f - f_q$ in the respective intervals, replace the upper limit f_q of the first integral by ∞, combine the integrals, and compare with (2.1–6) of Reference A)

Power spectrum of $\theta'(t) = W(f)$

$$\approx 4\pi^2f^2Q^{-2}[w(f_q+f) + w(f_q-f)] \tag{7.2}$$

This form is closely related to results customarily used in frequency modulation studies. It should be remembered that in (7.2) it is assumed that $0 < f \ll f_q$ and rms $I_N \ll Q$.

Additional terms in the approximation for $\Omega(\tau)$ may be obtained by expanding

$$\theta = \arc\tan \frac{I_s}{Q + I_c}$$

in descending powers of Q, multiplying two such series (one for time t and the other for time $t + \tau$) together, and averaging over t. If I_{c1}, I_{s1} and I_{c2}, I_{s2} denote the values of I_c, I_s at times t and $t + \tau$ respectively, the average values of the products of the I's may be obtained by expanding the characteristic function (obtainable from equation (7.5) given below by setting $z_5 = z_6 = z_7 = z_8 = 0$) of the four random variables $I_{c1}, I_{s1}, I_{c2}, I_{s2}$. This method is explained in Section 4.10 of Reference A. When $w(f)$ is symmetrical about f_q it is found that

[14] The form which we shall use is given by equation (2.1–5) of Reference A.

$$\begin{aligned}\overline{\theta_1\theta_2} &= \frac{g}{Q^2} + \frac{g^2}{Q^4} + \frac{8g^3}{3Q^6} + \cdots \\ \Omega(\tau) &= \overline{\theta_1'\theta_2'} = -\frac{d^2}{d\tau^2}\overline{\theta_1\theta_2} \\ &= -\frac{g''}{Q^2} - \frac{2}{Q^4}(gg'' + g'^2) - \frac{8}{Q^6}(g^2g'' + 2g'^2g) + \cdots\end{aligned} \tag{7.3}$$

From the exact expression for $\Omega(\tau)$ obtained below it is seen that the last equation in (7.3) is really asymptotic in character and the series does not converge. We infer that this is also true for the first equation of (7.3).

We shall now obtain the exact expression for the correlation function $\Omega(\tau)$ of $\theta'(t)$ when f_q is at the center of a symmetrical band of noise. At first sight it would appear that the easiest procedure is to calculate the correlation function for $\theta(t)$ and then obtain $\Omega(\tau)$ by differentiating twice. However, difficulties present themselves in getting θ outside the range $-\pi$, π since θ enters the expressions only as the argument of trigonometrical functions. Because I could not see any way to overcome this difficulty I was forced to deal with θ' directly. Unfortunately this increases the complexity since now the distribution of the time derivatives of I_c and I_s also must be considered.

We have

$$\tan\theta = \frac{I_s}{Q + I_c}, \qquad \sec^2\theta = 1 + \left(\frac{I_s}{Q + I_c}\right)^2$$

$$\theta' = \frac{(Q + I_c)I_s' - I_sI_c'}{\sec^2\theta(Q + I_c)^2} = \frac{(Q + I_c)I_s' - I_sI_c'}{(Q + I_c)^2 + I_s^2}$$

and the value of $\overline{\theta'(t)\theta'(t + \tau)}$ is the eight-fold integral

$$\begin{aligned}\Omega(\tau) = \int_{-\infty}^{+\infty} dI_{c1} \cdots \int_{-\infty}^{+\infty} dI_{s2}'\, p(I_{c1}, \cdots, I_{s2}') \\ \frac{(Q + I_{c1})I_{s1}' - I_{s1}I_{c1}'}{(Q + I_{c1})^2 + I_{s1}^2} \times \frac{(Q + I_{c2})I_{s2}' - I_{s2}I_{c2}'}{(Q + I_{c2})^2 + I_{s2}^2}\end{aligned} \tag{7.4}$$

where $p(I_{c1}, \cdots, I_{s2}')$ is an eight-dimensional normal probability density. As before, the subscripts 1 and 2 refer to times t and $t + \tau$, respectively. The most direct way of evaluating the integral (7.4) is to insert the expression for $p(I_{c1}, \cdots, I_{s2}')$ and then proceed with the integration. Indeed, this method was used the first time the integral (7.4) was evaluated. Later it was found that the algebra could be simplified by representing $p(I_{c1}, \cdots, I_{s2}')$ as the Fourier transform of its characteristic function. The second procedure will be followed here.

The characteristic function for I_{c1}, I_{c2}, I_{s1}, I_{s2}, I'_{c1}, I'_{c2}, I'_{s1}, I'_{s2} is, from (A2–2) and (A2–3) of Appendix II and Section 2.9 of Reference A,

$$\begin{aligned} &\text{ave. exp } i[z_1 I_{c1} + z_2 I_{c2} + z_3 I_{s1} + z_4 I_{s2} + z_5 I'_{c1} + z_6 I'_{c2} + z_7 I'_{s1} + z_8 I'_{s2}] \\ &\quad = \exp\left(-\tfrac{1}{2}\right) [b_0(z_1^2 + z_2^2 + z_3^2 + z_4^2) + b_2(z_5^2 + z_6^2 + z_7^2 + z_8^2) \\ &\qquad + 2b_1(z_1z_7 + z_2z_8 - z_3z_5 - z_4z_6) \\ &\qquad + 2g(z_1z_2 + z_3z_4) + 2g'(z_1z_6 - z_2z_5 + z_3z_8 - z_4z_7) \\ &\qquad - 2g''(z_5z_6 + z_7z_8) + 2h(z_1z_4 - z_2 z_3) \\ &\qquad + 2h'(z_1z_8 + z_2z_7 - z_3z_6 - z_4z_5) - 2h''(z_5z_8 - z_6z_7)]. \end{aligned} \tag{7.5}$$

Since we have included b_1, h, h', h'' this holds when f_q is not necessarily at the center of the noise band. However, henceforth we return to our assumption that f_q is placed at the center of a symmetrical noise band and take b_1, h, h', h'' to be zero.

The probability density of $I_{c1}, \cdots I'_{s2}$ which is to be placed in (7.4) is the eight-fold integral

$$\begin{aligned} p(I_{c1}, \cdots I'_{s2}) = (2\pi)^{-8} \int_{-\infty}^{+\infty} dz_1 \cdots \int_{-\infty}^{+\infty} dz_8 \\ \exp[-iz_1 I_{c1} - \cdots - iz_8 I'_{s2}] \times [\text{ch.f.}] \end{aligned} \tag{7.6}$$

where "ch.f." denotes the characteristic function obtained by setting b_1, h, h', h'' equal to zero on the right hand side of (7.5).

The integral (7.4) for $\Omega(\tau)$ may be written as

$$\Omega(\tau) = J_1 - J_2 - J_3 + J_4 \tag{7.7}$$

where J_1 is the 16-fold integral

$$\begin{aligned} J_1 = \int_{-\infty}^{+\infty} dI_{c1} \cdots \int_{-\infty}^{+\infty} dI'_{s2} (2\pi)^{-8} \int_{-\infty}^{+\infty} dz_1 \cdots \int_{-\infty}^{+\infty} dz_8 \\ \exp[-iz_1 I_{c1} - \cdots - iz_8 I'_{s2}] \\ \frac{(Q + I_{c1})(Q + I_{c2}) I'_{s1} I'_{s2}}{[(Q + I_{c1})^2 + I_{s1}^2][(Q + I_{c2})^2 + I_{s2}^2]} \times [\text{ch.f.}] \end{aligned} \tag{7.8}$$

and J_2, J_3, J_4 are obtained from J_1 by replacing the product $(Q + I_{c1})(Q + I_{c2})I'_{s1}I'_{s2}$ by $(Q + I_{c1})I'_{s1}I_{s2}I'_{c2}$, $I_{s1}I'_{c1}(Q + I_{c2})I'_{s2}$, $I_{s1}I'_{c1}I_{s2}I'_{c2}$ respectively.

The integration with respect to I_{c1} and I_{s1} in (7.8) may be performed at once. We replace $Q + I_{c1}$ and I_{s1} by x and y, respectively, and use

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, \frac{x}{x^2 + y^2} \, e^{-izx - i\zeta y} = \frac{-2\pi i z}{z^2 + \zeta^2}. \tag{7.9}$$

The integration with respect to I_{c2} and I_{s2} may be performed in a similar manner. In this way we obtain a 12-fold integral.

The integrations with respect to the I''s may be performed by using

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{+\infty} dI \int_{-\infty}^{+\infty} e^{-izI} f(z)\,dz &= f(0) \\ \frac{1}{2\pi}\int_{-\infty}^{+\infty} I\,dI \int_{-\infty}^{+\infty} e^{-izI} f(z)\,dz &= -i\left[\frac{df(z)}{dz}\right]_{z=0} \end{aligned} \tag{7.10}$$

The result is the four-fold integral

$$\begin{aligned} J_1 = (2\pi)^{-2}\int_{-\infty}^{+\infty} dz_1 \cdots \int_{-\infty}^{+\infty} dz_4 \frac{z_1 z_2(g'' - g'^2 z_3 z_4)}{(z_1^2 + z_3^2)(z_2^2 + z_4^2)} \\ \exp\left[-(b_0/2)(z_1^2 + z_2^2 + z_3^2 + z_4^2) - g(z_1 z_2 + z_3 z_4) + iQ(z_1 + z_2)\right]. \end{aligned} \tag{7.11}$$

In the same way J_2, J_3, J_4 may be reduced to the integrals obtained from (7.11) by replacing $z_1z_2(g'' - g'^2z_3z_4)$ by $-g'^2z_1^2z_4^2$, $-g'^2z_2^2z_3^2$ and $z_3z_4(g'' - g'^2z_1z_2)$, respectively. When the J's are combined in accordance with (7.7) we obtain an integral which may be obtained from (7.11) by replacing $z_1z_2(g'' - g'^2z_3z_4)$ by

$$g''(z_1z_2 + z_3z_4) + g'^2(z_1z_4 - z_2z_3)^2 \tag{7.12}$$

The terms $z_1^2 + z_3^2$ and $z_2^2 + z_4^2$ in the denominator may be represented as infinite integrals. Interchanging the order of integration and expressing (7.12) in terms of partial derivatives of an exponential function leads to the six-fold integral

$$\begin{aligned} \Omega(\tau) = (4\pi)^{-2}\int_0^\infty du \int_0^\infty dv \left[-g''\frac{\partial}{\partial g} + g'^2\frac{\partial^2}{\partial\alpha^2}\right]_{\alpha=0} \int_{-\infty}^{+\infty} dz_1 \cdots \int_{-\infty}^{+\infty} dz_4 \\ \exp\left[-(b_0 + u)(z_1^2 + z_3^2)/2 - (b_0 + v)(z_2^2 + z_4^2)/2 \right. \\ \left. -g(z_1z_2 + z_3z_4) - \alpha(z_1z_4 - z_2z_3) + iQ(z_1 + z_2)\right] \end{aligned} \tag{7.13}$$

where the subscript $\alpha = 0$ indicates that α is to be set equal to zero after the differentiations are performed.

When the four-fold integral in the z's is evaluated (7.13) becomes

$$\begin{aligned} \Omega(\tau) &= \int_0^\infty du \int_0^\infty dv \left[-g''\frac{\partial}{\partial g} + g'^2\frac{\partial^2}{\partial\alpha^2}\right]_{\alpha=0} \\ &\qquad \frac{1}{4D}\exp\left[-Q^2(2b_0 - 2g + u + v)/(2D)\right] \qquad (7.14) \\ &= \int_0^\infty du \int_0^\infty dv[(g'^2 - gg'')(2 - 2F + Q^2/g) - g'^2Q^2/g]e^{-F}/(4D_0^2) \end{aligned}$$

where

$$D = (b_0 + u)(b_0 + v) - g^2 - \alpha^2, \qquad F = Q^2(2b_0 - 2g + u + v)/(2D_0)$$

and D_0 denotes the value of D obtained by setting $\alpha = 0$. When differentiating with respect to α it is helpful to note that

$$\frac{\partial^2 f(D)}{\partial \alpha^2} = f''(D)\left(\frac{\partial D}{\partial \alpha}\right)^2 + f'(D)\,\frac{\partial^2 D}{\partial \alpha^2}$$

and that only $f'(D) = df/dD$ need be obtained since $\partial D/\partial \alpha$ vanishes when $\alpha = 0$.

In order to reduce the double integral to a single integral we make the change of variables

$$\begin{gathered} r = Q^2(b_0 + u - g)/(2D_0) \equiv \frac{Q^2(b_0 + u - g)}{2[(b_0 + u)(b_0 + v) - g^2]} \\ s = Q^2(b_0 + v - g)/(2D_0), \qquad F = r + s \\ \partial(r, s)/\partial(u, v) = -rs/D_0, \qquad 4srD_0 = Q^2[Q^2 - 2g(r + s)] \end{gathered} \tag{7.15}$$

The limits of integration for r and s are obtained by noting that the points $(0, 0)$, $(\infty, 0)$, (∞, ∞), $(0, \infty)$ in the (u, v) plane go into $(Q^2/(2b_0 + 2g), Q^2/(2b_0 + 2g))$, $(Q^2/(2b_0), 0)$, $(0, 0)$ $(0, Q^2/(2b_0))$, respectively, in the (r, s) plane. It may be verified that the region of integration in the (r, s) plane is the interior of the quadrilateral obtained by joining the above points by straight lines. Equation (7.14) may now be written as

$$\begin{aligned} \Omega(\tau) &= \int\!\!\int \left\{\frac{(g'^2 - gg'')(2 - 2r - 2s + Q^2/g) - g'^2 Q^2/g}{Q^2[Q^2 - 2g(r + s)]}\right\} e^{-r-s}\, dr\, ds \\ &= \frac{g'^2 - gg''}{2g^2}\, y_1 - \frac{g'^2}{2g^2}\, y_2 \end{aligned} \tag{7.16}$$

where y_1 and y_2 are the dimensionless quantities

$$y_1 = \int\!\!\int \frac{2g^2(2 - 2r - 2s + Q^2/g)}{Q^2[Q^2 - 2g(r + s)]}\, e^{-r-s}\, dr\, ds$$

$$y_2 = \int\!\!\int \frac{2ge^{-r-s}\, dr\, ds}{Q^2 - 2g(r + s)}$$

It is seen that

$$y_1 = 2gQ^{-2}\left\{y_2 + \int\!\!\int e^{-r-s}\, dr\, ds\right\}. \tag{7.17}$$

Since the integrands are functions of $r + s$ alone we are led to apply the transformation

$$\int\!\!\int_A f(r + s)\, dr\, ds = \int_0^\alpha u f(u)\, du + \int_\alpha^{2\beta} \frac{\alpha(2\beta - u)}{2\beta - \alpha} f(u)\, du \tag{7.18}$$

where A is the area enclosed by the quadrilateral whose vertices are at the points (r, s) given by $(0, 0)$, $(0, \alpha)$, (β, β), $(\alpha, 0)$ and it is assumed that β and α are positive. u is a new variable and is not the one introduced in (7.13).

Setting $\alpha = Q^2/(2b_0)$ and $\beta = Q^2/(2b_0 + 2g)$, using (7.18), and introducing the notation

$$\begin{aligned} \rho &= Q^2/(2b_0), & k &= g/b_0 \\ \xi &= Q^2/(2g) = \rho/k, & \lambda &= \frac{Q^2}{b_0 + g} = \frac{2\rho}{1 + k} \end{aligned} \tag{7.19}$$

permits us to write

$$\begin{aligned} \iint_A e^{-r-s}\,dr\,ds &= \int_0^\rho u e^{-u}\,du + \int_\rho^\lambda \frac{\rho(\lambda - u)}{\lambda - \rho} e^{-u}\,du \\ &= 1 - \frac{\lambda e^{-\rho}}{\lambda - \rho} + \frac{\rho e^{-\lambda}}{\lambda - \rho} \end{aligned} \tag{7.20}$$

and (7.17) yields

$$y_2 = \frac{\rho}{k} y_1 - 1 + \frac{2}{1 - k} e^{-\rho} - \frac{1 + k}{1 - k} e^{-2\rho/(1+k)} \tag{7.21}$$

where we have expressed λ in terms of ρ and k.

The double integral defining y_2 may be treated in the same way as (7.20):

$$y_2 = \iint_A \frac{e^{-r-s}\,dr\,ds}{\xi - r - s} = \int_0^\rho \frac{u e^{-u}\,du}{\xi - u} + \int_\rho^\lambda \frac{\rho(\lambda - u)e^{-u}\,du}{(\lambda - \rho)(\xi - u)}$$

Writing $u = \xi - (\xi - u)$ and $\lambda - u = \lambda - \xi + (\xi - u)$ in the two numerators leads to

$$\begin{aligned} y_2 = \xi \int_0^\rho \frac{e^{-u}\,du}{\xi - u} &- \int_0^\rho e^{-u}\,du \\ &- \xi \int_\rho^\lambda \frac{e^{-u}\,du}{\xi - u} + \frac{\rho}{\lambda - \rho} \int_\rho^\lambda e^{-u}\,du \end{aligned} \tag{7.22}$$

where we have used $\rho(\lambda - \xi)/(\lambda - \rho) = -\xi$ to simplify the coefficient of the third integral. When the second and fourth integrals are evaluated, their contribution to y_2 is found to be equal to the terms independent of y_1 on the right of (7.21). Hence, comparison of equations (7.21) and (7.22) shows that

$$y_1 = \int_0^\rho \frac{e^{-u}\,du}{\xi - u} - \int_\rho^\lambda \frac{e^{-u}\,du}{\xi - u} \tag{7.23}$$

The integrals in (7.23) may be evaluated in terms of the exponential integral $Ei(x)$ defined by, for x real,

$$Ei(x) = \int_{-\infty}^{x} e^t \, dt/t = C + \tfrac{1}{2} \log_e x^2 + \sum_{n=1}^{\infty} \frac{x^n}{n!\,n}$$
$$\sim e^x \sum_{n=0} n!/x^{n+1}$$

where $C = .577 \cdots$ is Euler's constant and Cauchy's principal value of the integral is to be taken when $x > 0$. We set $t = \xi - u$ and obtain

$$y_1 = e^{-\rho/k} \left\{ Ei[\rho/k] - 2Ei[\rho(1 - k)/k] + Ei\left[\frac{\rho(1 - k)}{k(1 + k)}\right]\right\}$$

where we have again expressed ξ and λ in terms of ρ and k.

A power series for y_1 which converges when $-1/3 \leq k < 1$ may be obtained by expanding the denominators of the integrands in (7.23) in powers of u/ξ and integrating termwise:

$$\begin{aligned} y_1 = {} & \xi^{-1}[1 - 2e^{-\rho} + e^{-\lambda}] \\ & + 1!\xi^{-2}[1 - 2(1 + \rho/1!)e^{-\rho} + (1 + \lambda/1!)e^{-\lambda}] \\ & + 2!\xi^{-3}[1 - 2(1 + \rho/1! + \rho^2/2!)e^{-\rho} + (1 + \lambda/1! + \lambda^2/2!)e^{-\lambda}] \\ & + \cdots \end{aligned} \tag{7.24}$$

The following special values may be obtained from the equation given above. When $\rho = 0$

$$\begin{aligned} y_1 &= -\log_e (1 - k^2) \\ y_2 &= 0 \end{aligned} \tag{7.25}$$

This result may also be obtained by evaluating the integral obtained when we set $Q = 0$, $z_1 = r_1 \cos \theta_1$, $z_3 = r_1 \sin \theta_1$, $z_2 = r_2 \cos \theta_2$, $z_4 = r_2 \sin \theta_2$ in (7.11) and (7.12).

Near $k = 1$,

$$\begin{aligned} y_1 &\approx e^{-\rho}[Ei(\rho) - C - \log_e \rho(1 - k^2)] \\ y_2 &\approx \rho y_1 - 1 + (1 + \rho)e^{-\rho} \end{aligned} \tag{7.26}$$

Near $k = 0$,

$$y_1 \approx k(1 - e^{-\rho})^2/\rho, \qquad y_2 \approx y_1 \tag{7.27}$$

except when $\rho = 0$ in which case y_1 is approximately k^2.

When ρ is large

$$\begin{aligned} y_1 &\sim \frac{k}{\rho} + \frac{1!k^2}{\rho^2} + \frac{2!k^3}{\rho^3} + \frac{3!k^4}{\rho^4} + \cdots \\ y_2 &\sim -1 + \frac{\rho}{k} y_1 \sim \frac{1!k}{\rho} + \frac{2!k^2}{\rho^2} + \cdots \end{aligned} \tag{7.28}$$

except near $k = 1$ where both y_1 and y_2 have logarithmic infinities. The asymptotic expansion (7.3) for $\Omega(\tau)$, which was obtained by the first method of this section, may he checked by inserting (7.28) in the expression (7.16) for $\Omega(\tau)$ in terms of y_1 and y_2.

Values of y_1 and y_2 tabulated as functions of k for various values of ρ are given in Table 3. Negative values of k have not been considered since they

TABLE 3

VALUES OF y_1 AND y_2 USED IN COMPUTATION OF CORRELATION FUNCTION OF $d\theta/dt$

$$\Omega(\tau) = \overline{\theta'(t)\theta'(t+\tau)} = [g'^2(y_1 - y_2) - gg''y_1]/(2g^2)$$

$$g \equiv g(\tau) = \int_0^\infty w(f) \cos 2\pi(f - f_q)\tau\, df, \quad k = g(\tau)/g(0)$$

k	Values of y_1					Values of y_2			
	ρ					ρ			
	0	.5	1	2	5	.5	1	2	5
0	0	0	0	0	0	0	0	0	0
.1	.01005	.03526	.04224	.03854	.02000	.03171	.04147	.03936	.02051
.2	.04082	.08043	.09003	.07979	.04105	.06550	.08654	.08275	.04283
.3	.09431	.1379	.1452	.1246	.06292	.1022	.1363	.1315	.06702
.4	.1744	.2110	.2102	.1740	.08586	.1432	.1926	.1870	.09384
.5	.2877	.3056	.2886	.2296	.1101	.1914	.2579	.2515	.1238
.6	.4463	.4278	.3860	.2942	.1358	.2481	.3368	.3289	.1576
.7	.6733	.5953	.5129	.3721	.1636	.3220	.4379	.4269	.1975
.8	1.0216	.8416	.6914	.4729	.1941	.4275	.5803	.5602	.2461
.84	1.2228	.9798	.7888	.5242	.2075	.4866	.6593	.6318	.2693
.88	1.4890	1.1590	.9127	.5866	.2219	.5641	.7619	.7226	.2964
.90	1.6607	1.2742	.9898	.6241	.2296	.6138	.8260	.7752	.3114
.92	1.8734	1.4144	1.0834	.6686	.2378	.6753	.9058	.8486	.3294
.94	2.1507	1.5948	1.2024	.7217	.2466	.7550	1.0093	.9333	.3498
.96	2.5459	1.8486	1.3668	.7939	.2566	.8711	1.1558	1.0546	.3752
.97	2.8285	2.0251	1.4815	.8414	.2623	.9474	1.2605	1.1366	.3849
.98	3.2289	2.2762	1.6405	.9073	.2690	1.0704	1.4081	1.2548	.4119
.99	3.9170	2.7080	1.9066	1.0127	.2778	1.2773	1.6610	1.4505	.4429
.995	4.6072	3.1341	2.1721	1.1125	.2846	1.4838	1.9175	1.6416	.4705
.997	5.1175	3.4445	2.3622	1.1866	.2889	1.6367	2.1048	1.7859	.4893

are not required for the case in which I_N has a normal law power spectrum, the case discussed in the next section.

8. POWER SPECTRUM OF $\frac{d\theta}{dt}$ WHEN I_N HAS NORMAL LAW POWER SPECTRUM

The problem of computing the power spectrum $W(f)$ of $\theta'(t)$ appears to be a difficult one.* In order to obtain an answer without an excessive amount of work we have had to do two things which are rather restrictive. First, we confine our attention to the case in which the power spectrum $w(f)$ of

*Since the above was written the general f. m. problem has been studied by D. Middleton. He generalizes our (7.11) and (7.12), introduces polar coordinates, expands the integrand in powers of g, and integrates termwise. $W(f)$ then follows somewhat as in a.m. theory.

I_N is of the normal law type (our method could be applied to other types but g' and g'' would be more complicated functions of τ and Table 3 would have to be extended to negative values of k, if they should occur). Second, we resort to numerical integration to obtain a portion of $W(f)$. Because of the second item our results are either tabulated or are given as curves, shown in Figs. 8 and 9, except when $Q = 0$ (noise only) in which case the power spectrum of θ' is given by the series (8.7).

The power spectrum of I_N is assumed to be

$$w(f) = \frac{\psi_0}{\sigma\sqrt{2\pi}}\, e^{-(f-f_q)^2/(2\sigma^2)} \tag{8.1}$$

The mean square value of I_N is equal to that of a noise current whose power spectrum has the constant value of $\psi_0/(\sigma\sqrt{2\pi})$ over a band of width $f_b - f_a = \sigma\sqrt{2\pi} = \sigma 2.507$. The value of $w(f)$ is one quarter of its mid-band value at the points $f - f_q = \pm\sigma\sqrt{2 \log_e 4} = \pm\sigma 1.665$ (the 6 *db* points) and the distance between these points is 3.330σ. Integration of (8.1) shows that the mean square value of I_N is ψ_0 in accordance with our customary notation. The mid-band value of $w(f)$ is $\psi_0/(\sigma\sqrt{2\pi})$.

Assuming $f_q \gg \sigma$ and evaluating the integrals (A2–1) of Appendix II defining b_0 and g gives

$$\begin{aligned} &b_0 = \psi_0, \qquad g = \psi_0 e^{-2(\pi\sigma\tau)^2} = \psi_0 e^{-u^2/2} \\ &\qquad g'/g = -uu' = -2\pi\sigma u, \qquad g''/g = -(2\pi\sigma)^2(1 - u^2) \\ &\qquad\qquad \frac{g''g - g'^2}{g^2} = -(2\pi\sigma)^2, \qquad k = g/b_0 = e^{-u^2/2} \end{aligned} \tag{8.2}$$

where we have set

$$u = 2\pi\sigma\tau, \qquad u' = 2\pi\sigma \tag{8.3}$$

and the primes on g and u denote differentiation with respect to τ. The correlation function is accordingly, from (7.16).

$$\Omega(\tau) = 2\pi^2\sigma^2(y_1 - u^2 y_2) \tag{8.4}$$

If $\theta'(t)$ be regarded as a noise current its power spectrum is

$$W(f) = 4 \int_0^\infty \Omega(\tau) \cos 2\pi f\tau \, d\tau \tag{8.5}$$

When noise alone is present, ρ is zero and (7.25) yields

$$\Omega(\tau) = -2\pi^2\sigma^2 \log_e (1 - k^2) = -2\pi^2\sigma^2 \log_e (1 - e^{-u^2}) \tag{8.6}$$

In this case the power spectrum is, from (8.3), (8.5), and (8.6),

$$
\begin{aligned}
W_N(f) &= -4\pi\sigma \int_0^\infty \cos(uf/\sigma) \log(1 - e^{-u^2})\, du \\
&= 2\sigma\pi^{3/2} \sum_{n=1}^{\infty} n^{-3/2} e^{-f^2/(4n\sigma^2)},
\end{aligned} \tag{8.7}
$$

the series being obtained by expanding the logarithm and integrating termwise. When this equation was used for computation it was found convenient to apply the Euler summation formula to sum the terms in the series beyond the (N − 1)st. Writing b for $f^2/(4\sigma^2)$, the series in (8.7) becomes

$$
\begin{aligned}
&1^{-3/2}e^{-b/1} + 2^{-3/2}e^{-b/2} + \cdots (N-1)^{-3/2}e^{-b/(N-1)} \\
&\quad + (\pi/b)^{1/2} \operatorname{erf}[(b/N)^{1/2}] + N^{-3/2}e^{-b/N}\left[\frac{1}{2} - \frac{1}{12N}\left(-\frac{3}{2} + \frac{b}{N}\right)\right. \\
&\qquad \left. + \frac{1}{720N^3}\left(-\frac{105}{8} + \frac{105}{4}\frac{b}{N} - \frac{21}{2}\frac{b^2}{N^2} + \frac{b^3}{N^3}\right) + \cdots\right]
\end{aligned} \tag{8.8}
$$

When b is zero the sum[15] of the series is 2.61237 $\cdots$. The values for $\rho = 0$ in Table 4 were computed by taking $N = 12$ in (8.8). As $b \to \infty$ the dominant term in (8.8) is seen to be the one containing erf (choose N so that $b = N^{3/2}$). Hence as $f \to \infty$

$$
W_N(f) \sim 4\pi^2\sigma^2/f. \tag{8.9}
$$

When both noise and the sine wave are present it is convenient to split the power spectrum into three parts. The first part, $W_1(f)$, is proportional to $W_N(f)$, the power spectrum with noise alone. The second part $W_2(f)$ is proportional to the form $W(f)$ assumes when rms $I_N \ll Q$ and the third part $W_3(f)$ is of the nature of a correction term. This procedure is suggested when we subtract the leading terms in the expressions (7.26) and (7.27) (corresponding to $k = 1$ and $k = 0$, respectively) from y_1. Likewise we subtract the leading term in y_2, (7.27), at $k = 0$ but do not bother to do so at the end $k = 1$ because u^2y_2 approaches zero there. We therefore write

$$
\begin{aligned}
y_1 - u^2y_2 &= [y_1 + e^{-\rho}\log(1 - k^2) - k(1 - e^{-\rho})^2/\rho - u^2y_2 \\
&\quad + u^2k(1 - e^{-\rho})^2/\rho] - e^{-\rho}\log(1 - k^2) + (1 - u^2)k(1 - e^{-\rho})^2/\rho \\
&= Z(u) - e^{-\rho}\log(1 - k^2) - \frac{g''(2\pi\sigma)^{-2}}{b_0\rho}(1 - e^{-\rho})^2
\end{aligned} \tag{8.10}
$$

[15] "Theory and Application of Infinite Series," Knopp, (1928), page 561.

where $Z(u)$ denotes the function enclosed by the brackets in the first equation and the expressions for g''/g and k in (8.2) have been used in the replacement of $(1 - u^2)k$.

TABLE 4

VALUES OF $W_3(f)/(4\pi^2\sigma)$

$\frac{6f}{\sigma\pi}$	$\rho = 0$	0.5	1.0	2.0	5.0
0	0	−.03517	−.03891	−.02444	−.001948
1	0	−.03003	−.03196	−.01830	−.001814
2	0	−.01717	−.01486	−.003304	.004052
3	0	−.002436	.004014	.01252	.008225
4	0	.008757	.01730	.02244	.01027
6	0	.01478	.02157	.02167	.007665
8	0	.01018	.01366	.01237	.003505
10	0	.005768	.007378	.006201	.001437
12	0	.004027	.004463	.003552	.0006439

VALUES OF $W(f)/(4\pi^2\sigma)$

$\frac{6f}{\sigma\pi}$	$\rho = 0$	0.5	1.0	2.0	5.0
0	.7369	.4118	.2322	.07529	.003017
1	.7098	.4294	.2672	.1134	.02342
2	.6439	.4516	.3231	.1784	.05828
3	.5542	.4225	.3225	.1947	.06852
4	.4623	.3496	.2654	.1580	.01590
6	.3195	.2178	.1508	.07554	.01540
8	.2390	.1553	.1019	.04506	.005325
10	.1908	.1215	.07768	.03206	.002726
12	.1595	.1003	.06306	.02511	.001719

Inserting (8.10) in the expression (8.4) for $\Omega(\tau)$ and taking the Fourier transform (8.5) leads to

$$\begin{aligned}
W(f) &= W_1(f) + W_2(f) + W_3(f) \\
W_1(f) &= e^{-\rho} W_N(f) \\
W_2(f) &= -\frac{(1 - e^{-\rho})^2}{2b_0\rho} \int_0^\infty g'' \cos 2\pi f\tau \, d\tau \\
&= \frac{(1 - e^{-\rho})^2}{\rho} (2\pi f)^2 \frac{e^{-f^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}} \\
W_3(f) &= 4\pi\sigma \int_0^\infty Z(u) \cos (uf/\sigma) \, du
\end{aligned} \tag{8.11}$$

In these equations $W_N(f)$ is obtained from (8.7), and $W_2(f)$ by two-fold integration by parts to reduce g'' to g then evaluating the integral obtained

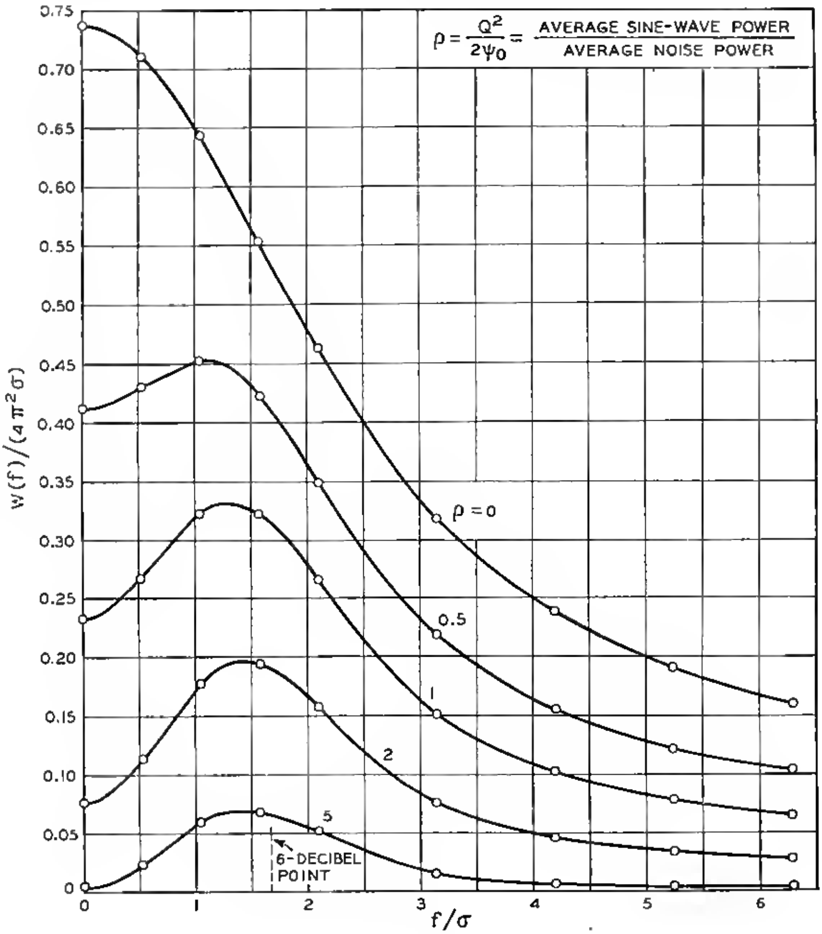

Fig. 8—Power spectrum of $d\theta/dt$.

Power spectrum of I_N is assumed to be

$$\psi_0(\sigma\sqrt{2\pi})^{-1} \exp\left[-(f - f_q)^2/(2\sigma^2)\right].$$

In this expression f is a frequency near f_q. The f in $W(f)$ and in the abscissa is a much lower frequency. $W(f)$ = power spectrum of $\theta' = d\theta/dt$, θ' being regarded as a random noise current. Dimensions of $W(f)\,df$ same as $(d\theta/dt)^2$ or (radians)^2/sec.^{2}.

by substituting the expression (8.2) for g. That $W(f)$ approaches $W_2(f)$ as $\rho \to \infty$ follows when expression (8.11) for $W_2(f)$ is compared with the limiting form (8.13) given below.

Instead of dealing with $W(f)$ it is more convenient to deal with $(4\pi^2\sigma)^{-1}$ $W(f)$ which is the sum of the three components

$$
\begin{aligned}
(4\pi^2\sigma)^{-1}W_1(f) &= \frac{e^{-\rho}}{2\sqrt{\pi}} \sum_{n=1}^{\infty} n^{-3/2} e^{-f^2/(4n\sigma^2)} \\
(4\pi^2\sigma)^{-1}W_2(f) &= \frac{(1-e^{-\rho})^2}{\rho\sqrt{2\pi}} \left(\frac{f}{\sigma}\right)^2 e^{-f^2/(2\sigma^2)} \qquad (8.12) \\
(4\pi^2\sigma)^{-1}W_3(f) &= \frac{1}{\pi}\int_0^{\infty} Z(u) \cos(uf/\sigma)\, du
\end{aligned}
$$

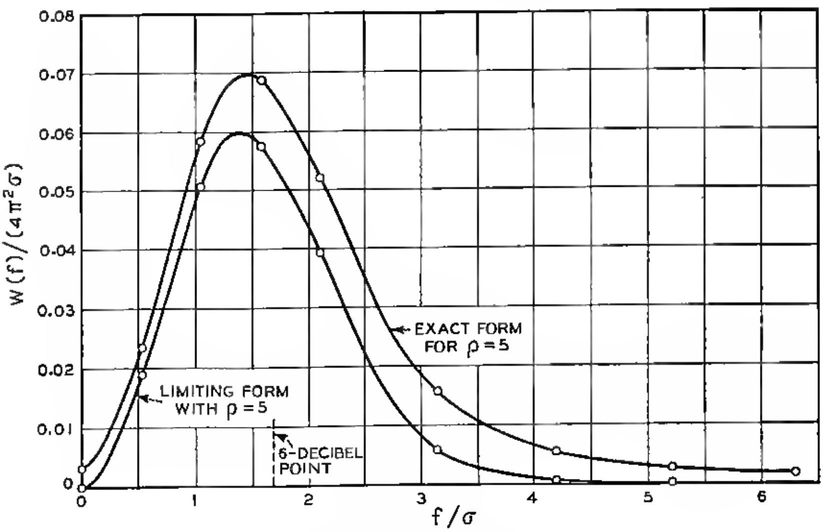

Fig. 9—Approach of $W(f)$ to limiting form.
As $\rho \to \infty$, $W(f) \to 4\pi^2\sigma\,(\rho\sqrt{2\pi})^{-1}\,(f/\sigma)^2 \exp[-f^2/(2\sigma^2)]$.

The integral involving $Z(u)$ has been computed by Simpson's rule, y_1 and y_2 being obtained from Table 3, with the results shown in the first section of Table 4. The value of $W_2(f)$ may be computed directly, and $W_1(f)$ may be obtained from $W_N(f)$. The values of these two functions together with those of W_3 (f) enable us to compute the values of $(4\pi^2\sigma)^{-1}W(f)$ given in Table 4 and plotted in Fig. 8.

Since, as is shown by (8.9), $W_N(f)$ varies as $1/f$ for large values of f, the areas under the curves of Fig. 8 become infinite. This agrees with the fact that the mean square value of θ' is infinite.

The values of $(4\pi^2\sigma)^{-1}W(0)$ for ρ equal to 0, .5, 1, 2, and 5 are .7369, 4118, 2322, .07529, and .003017 respectively. When these values are plotted on semi-log paper they tend to lie on a straight line whose slope suggests that $W(0)$ decreases as $e^{-\rho}$ when ρ becomes large.

The limiting form assumed by $W(f)$ as $\rho \to \infty$ is given by equation (7.2). When the normal law expression (8.1) assumed in this section for the power spectrum of I_N is put in (7.2) we find that

$$W(f) \to \frac{4\pi^2\sigma}{\rho\sqrt{2\pi}} \left(\frac{f}{\sigma}\right)^2 e^{-f^2/(2\sigma^2)} \tag{8.13}$$

Fig. 9 shows that for $\rho = 5$ the limiting form (8.13) agrees quite well with the exact form computed above.

Both (7.2) and (8.13) show that, for small values of f, the power spectrum of θ' varies as f^2 when $\rho >> 1$. This is in accord with Crosby's* result that the voltage spectrum of the random noise in the output of a frequency modulation receiver is triangular when the carrier to noise ratio is large. When this ratio becomes small he finds that the spectrum becomes rectangular. Fig. 8 shows this effect in that the areas under the curves between the ordinates at $f = 0$ and $f = \lambda\sigma$ (where λ is some number, generally less than unity, depending on the ratio of the widths of the i.f. and audio bands) become rectangles, approximately, as ρ decreases.

APPENDIX I

The Integral $Ie(k, x)$

The integral[16]

$$Ie(k,x) = \int_0^x e^{-u} I_0(ku)\, du, \tag{A1-1}$$

where $I_0(ku)$ denotes the Bessel function of imaginary argument and order zero, occurs in Sections 2 and 6. The following special cases are of interest.

$$\begin{aligned} Ie(0,\ x) &= 1 - e^{-x} \\ Ie(1,\ x) &= xe^{-x}[I_0(x) + I_1(x)] \\ Ie(k,\ \infty) &= \frac{1}{\sqrt{1-k^2}} \end{aligned} \tag{A1-2}$$

The second of these relations is due to Bennett.[17]

* M. G. Crosby, "Frequency Modulation Noise Characteristics," Proc. I. R. E. Vol. 25 (1937), 472–514. See also J. R. Carson and T. C. Fry, "Variable Electric Circuit Theory with Application to the Theory of Frequency Modulation," B.S.T.J. Vol. 16 (1937), 513–540.

[16] The notation was chosen to agree with that used by Bateman and Archibald (Guide to Tables of Bessel Functions appearing in "Math. Tables and Aids to Comp.", Vol. 1 (1944) pp. 205–308) to discuss integrals used by Schwarz (page 248).

[17] It is given in equation (62) of the reference cited in connection with our equation (1.2) in Section 1.

The values in the table given below were computed by Simpson's rule for numerical integration. The work was checked at several points by using

$$Ie(k, x) = \sum_{n=0}^{\infty} (k/2)^{2n} \frac{(2n)!}{n!n!} A_n$$

where

$$A_n = 1 - \left[1 + x + \frac{x^2}{2!} \cdots + \frac{x^{2n}}{(2n)!}\right] e^{-x}$$

When x is so large that $Ie(k, x)$ is nearly equal to $Ie(k, \infty)$ we have

$$Ie(k, x) \sim (1 - k^2)^{-1/2} - [2k(1 - k)]^{-1/2}(2/\sqrt{\pi}) \int_{t_1}^{\infty} e^{-t^2}\, dt$$

where $t_1 = \sqrt{x(1 - k)}$. However, this was not found to be especially useful in checking the values given in the table.

TABLE OF $Ie(k, x) = \int_0^x e^{-u} I_0(ku)\, du$

x	k						
	0	.2	.4	.6	.8	.9	1.0
0	0	0	0	0	0		0
.2	.1813	.1813	.1814	.1815	.1816		.1818
.4	.3297	.3298	.3303	.3311	.3322		.3337
.6	.4512	.4517	.4530	.4554	.4586		.4629
.8	.5507	.5516	.5545	.5593	.5661		.5749
1.0	.6321	.6337	.6386	.6468	.6584		.6736
.2	.6988	.7012	.7086	.7209	.7386		.7620
.4	.7534	.7567	.7669	.7841	.8089		.8422
.6	.7981	.8025	.8157	.8383	.8712		.9157
.8	.8347	.8401	.8566	.8850	.9267		.9839
2.0	.8647	.8712	.8910	.9255	.9766		1.0476
.2	.8892	.8968	.9201	.9607	1.0217		1.1075
.4	.9093	.9179	.9446	.9916	1.0627		1.1642
.6	.9257	.9354	.9655	1.0186	1.1001		1.2183
.8	.9392	.9499	.9831	1.0424	1.1345		1.2699
3.0	.9502	.9618	.9982	1.0635	1.1661		1.3195
.2	.9592	.9718	1.0110	1.0822	1.1953		1.3672
.4	.9666	.9800	1.0220	1.0988	1.2223		1.4132
.6	.9727	.9868	1.0314	1.1136	1.2475		1.4578
.8	.9776	.9925	1.0394	1.1268	1.2708		1.5010
4.0	.9817	.9971	1.0463	1.1386	1.2926		1.5430
.2	.9830	1.0010	1.0522	1.1492	1.3130		1.5839
.4	.9877	1.0043	1.0574	1.1587	1.3320		1.6237
.6	.9899	1.0070	1.0619	1.1672	1.3499		1.6625
.8	.9918	1.0092	1.0657	1.1749	1.3666		1.7005
5.0	.9933	1.0111	1.0690	1.1818	1.3823		1.7376
5.4	.9955	1.0140	1.0743	1.1937	1.4110		1.8095

TABLE—*Continued*

x	k						
	0	.2	.4	.6	.8	.9	1.0
5.8	.9970	1.0160	1.0783	1.2034	1.4364		1.8786
6.2	.9980	1.0174	1.0814	1.2114	1.4590		1.9452
6.6	.9986	1.0183	1.0837	1.2180	1.4792		2.0097
7.0	.9991	1.0190	1.0854	1.2234	1.4972		2.0722
7.4	.9994	1.0195	1.0867	1.2278	1.5134		2.1328
7.8	.9996	1.0198	1.0876	1.2375	1.5279		2.1917
8.2	.9997	1.0201	1.0885	1.2346	1.5409		2.2491
8.6	.9998	1.0202	1.0891	1.2371	1.5526		2.3050
9.0	.9999	1.0203	1.0896	1.2393	1.5631		2.3597
10.0	1.0000	1.0205	1.0902	1.2431	1.5852	1.9207	2.4910
11.0	1.0000	1.0206	1.0907	1.2456	1.6024	1.9668	2.6157
12.0	1.0000	1.0206	1.0909	1.2471	1.6158	2.0066	2.7347
13.0	1.0000	1.0206	1.0910	1.2482	1.6263	2.0411	2.8487
14.0	1.0000	1.0206	1.0910	1.2488	1.6346	2.0711	2.9584
15.0	1.0000	1.0206	1.0911	1.2492	1.6412	2.0973	3.0641
∞	1.0000	1.0206	1.0911	1.2500	1.6667	2.2942	∞

x	k			
	.86	.90	.96	1.0
15.0	1.8773	2.0973	2.5810	3.0641
16.0	1.8899	2.1201	2.6371	3.1663
17.0	1.9006	2.1403	2.6894	3.2653
18.0	1.9095	2.1579	2.7381	3.3614
19.0	1.9171	2.1737	2.7837	3.4548
20.0	1.9235	2.1870	2.8263	3.5457
∞	1.9597	2.2942	3.5714	∞

APPENDIX II

SECOND MOMENTS ASSOCIATED WITH I_c AND I_s

The in-phase and quadrature components of the noise current I_N

$$
\begin{aligned}
I_c(t) &= \sum_{n=1}^{M} c_n \cos\left[(\omega_n - q)t - \varphi_n\right] \\
I_s(t) &= \sum_{n=1}^{M} c_n \sin\left[(\omega_n - q)t - \varphi_n\right]
\end{aligned}
\tag{3.3}
$$

are closely related to the envelope R and phase angle θ of the total current, this relationship being being shown by the equations (3.4) and (3.5). $I_c(t)$ and $I_s(t)$ and their time derivatives may be regarded as random variables. In much of our work we have to deal with the probability distribution of these random variables. By virtue of the representation (3.3) and the central limit theorem[18] this distribution is normal in the several variables. The coefficients in the quadratic form occurring in the exponent are deter-

[18] Section 2.10 of Reference A.

mined by the second moments of the variables.[19] Here we state these moments. Some of the moments have already been given in Sections 3.7 and 3.8 of Reference A. For the sake of completeness we shall also give them here. The new results given below are derived in much the same way as those given in Reference A.

Let

$$
\begin{aligned}
b_n &= (2\pi)^n \int_0^\infty w(f)(f - f_q)^n \, df \\
b_0 &= \int_0^\infty w(f) \, df = \psi_0 \\
g &= \int_0^\infty w(f) \cos 2\pi(f - f_q)\tau \, df \\
h &= \int_0^\infty w(f) \sin 2\pi(f - f_q)\tau \, df
\end{aligned}
\tag{A2-1}
$$

and let g', g'', h', h'' denote the first and second derivatives of g and h with respect to τ. For example,

$$
g' = -2\pi \int_0^\infty w(f)(f - f_q) \sin 2\pi(f - f_q)\tau \, df
$$

Incidentally, in many of our cases $w(f)$ is assumed to be symmetrical about f_q. This introduces considerable simplification because b_1, b_3, b_5, $\cdots$, h, h', h'', reduce to zero.

The following table gives values of b_n's and g for two cases of frequent occurrence

	Ideal band pass filter centered on f_q	Normal law filter centered on f_q, $f_q \gg \sigma$
$w(f)$	w_0 for $f_a < f < f_b$ and zero elsewhere	$\dfrac{\psi_0}{\sigma\sqrt{2\pi}} e^{-(f-f_q)^2/2\sigma^2}$
b_0	$w_0(f_b - f_a)$	ψ_0
b_2	$\pi^2 w_0(f_b - f_a)^3/3$	$4\pi^2\sigma^2\psi_0$
b_4	$\pi^4 w_0(f_b - f_a)^5/5$	$48\pi^4\sigma^4\psi_0$
g	$(\pi\tau)^{-1} w_0 \sin \pi(f_b - f_a)\tau$	$\psi_0 e^{-2(\pi\sigma\tau)^2}$

If we write I_c, I_c', I_c'' for $I_c(t)$, $I_c'(t)$, $I_c''(t)$, where the primes denote differ-

[19] Section 2.9 of Reference A.

entiation with respect to t, and do the same for $I_s(t)$ and its derivatives we have, from Section 3.8 of Reference A,

$$
\begin{aligned}
\overline{I_c^2} &= \overline{I_s^2} = b_0, \qquad \overline{I_cI_s} = 0 \\
\overline{I_cI_s'} &= -\overline{I_c'I_s} = b_1, \qquad \overline{I_cI_c'} = \overline{I_sI_s'} = 0 \\
\overline{I_c'^2} = \overline{I_s'^2} &= -\overline{I_cI_c''} = -\overline{I_sI_s''} = b_2, \qquad \overline{I_c'I_s'} = \overline{I_cI_s''} = \overline{I_sI_c''} = 0 \\
\overline{I_c'I_s''} &= -\overline{I_s'I_c''} = b_3, \qquad \overline{I_c'I_c''} = \overline{I_s'I_s''} = 0 \\
\overline{I_c''^2} &= \overline{I_s''^2} = b_4, \qquad \overline{I_c''I_s''} = 0
\end{aligned}
\tag{A2-2}
$$

When we deal with moments in which the arguments of the two variables are separated by an interval τ as in (see the last of equations (3.7–11) of Reference A)

$$\overline{I_c(t)I_s(t+\tau)} = h,$$

it is convenient to denote the argument t by the subscript 1 and the argument $t+\tau$ by 2. Then our example becomes

$$\overline{I_{c1}I_{s2}} = h$$

We shall need the following moments of this type.

$$
\begin{aligned}
\overline{I_{c1}I_{c2}} = \overline{I_{s1}I_{s2}} &= g, \qquad \overline{I_{c1}I_{s2}} = -\overline{I_{c2}I_{s1}} = h \\
\overline{I_{c1}I_{c2}'} = \overline{I_{s1}I_{s2}'} &= -\overline{I_{c1}'I_{c2}} = -\overline{I_{s1}'I_{s2}} = g' \\
\overline{I_{c1}I_{s2}'} = \overline{I_{c2}I_{s1}'} &= -\overline{I_{c1}'I_{s2}} = -\overline{I_{c2}'I_{s1}} = h' \\
\overline{I_{c1}'I_{c2}'} = \overline{I_{s1}'I_{s2}'} &= -g'', \qquad \overline{I_{c1}'I_{s2}'} = -\overline{I_{c2}'I_{s1}'} = -h''
\end{aligned}
\tag{A2-3}
$$

It should be remembered that in these equations the primes on the I's denote differentiation with respect to t while the primes on g and h denote differentiation with respect to τ.

APPENDIX III

Evaluation of a Multiple Integral

Several multiple integrals encountered during the preparation of this paper were initially evaluated by the following procedure. The integral was first converted into a multiple series by expanding a portion of the integrand and integrating termwise. It was found possible to sum these series when one of the factorials in the denominator was represented as a contour integral. This reduced the multiple integral to a contour integral and sometimes the latter could be evaluated.

We shall illustrate this procedure by examining the integral

$$I = \int_{-\pi}^{\pi} d\theta \int_0^\infty dx\, x \exp\left[-x^2 + 2a \cos\theta + 2bx \sin\theta + c \sin^2\theta\right] \tag{A3-1}$$

Expanding that part of the exponential which contains the trigonometrical terms and integrating termwise gives

$$I = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{a^{2n} b^{2m} c^{\ell} \pi \Gamma(\ell + m + \frac{1}{2})}{n!\,\ell!\,(\ell + m + n)!\,\Gamma(m + \frac{1}{2})}$$

where we have used

$$2^{2n}\,\Gamma(n + \tfrac{1}{2})n! = \sqrt{\pi}(2n)!$$

We next make the substitution

$$\frac{1}{(\ell + m + n)!} = \frac{1}{2\pi i} \int_C \frac{e^t\, dt}{t^{\ell+m+n+1}} \tag{A3-2}$$

where the path of integration C is a circle chosen large enough to ensure the convergence of the series obtained when the order of summation and integration is changed. The summations may now be performed:

$$\begin{aligned} I &= \frac{1}{2i} \int_C dt\, e^{t + a^2/t} \sum_{m=0}^{\infty} b^{2m} t^{-m-1} (1 - ct^{-1})^{-m-1/2} \\ &= \frac{1}{2i} \int_C \frac{t^{-1/2}(t - c)^{1/2}}{t - c - b^2}\, e^{t + a^2/t}\, dt \end{aligned} \tag{A3-3}$$

C encloses the pole at $c + b^2$ and the branch point at c as well as the origin.

When a^2 is zero the integral may be reduced still further. Let c be complex and b such that the point $c + b^2$ does not lie on the line joining 0 to c. Deform C until it consists of an isolated loop about $c + b^2$ and a loop about 0 and c, the latter consisting of small circles about 0 and c joined by two straight portions running along the line joining 0 to c. The contributions of the small circles about 0 and c vanish in the limit. Along the portion starting at 0 and running to c, $\arg(t - c) = -\pi + \arg c$, and along the portion starting at c and running to 0, $\arg(t - c) = \pi + \arg c$. On both portions $\arg t = \arg c$. Bearing this in mind and setting $t = c \sin^2\theta$ on the two portions gives

$$I_{a=0} = \pi b(c + b^2)^{-1/2} e^{c+b^2} + 2c \int_0^{\pi/2} \frac{\cos^2\theta\, e^{c \sin^2\theta}}{b^2 + c\cos^2\theta}\, d\theta \tag{A3-4}$$

The integral may be expressed in terms of the function

$$Ie(k, x) = \int_0^x e^{-u} I_0(ku)\, du$$

by noting that

$$\int_0^\pi \frac{e^{-\alpha-\beta\cos v}}{\alpha+\beta\cos v}\,dv = \int_0^\pi dv\left[\frac{1}{\alpha+\beta\cos v} - \int_0^1 e^{-t(\alpha+\beta\cos v)}\,dt\right]$$

$$= \pi(\alpha^2-\beta^2)^{-1/2} - \pi\int_0^1 e^{-\alpha t} I_0(\beta t)\,dt \tag{A3-5}$$

$$= \pi(\alpha^2-\beta^2)^{-1/2} - (\pi/\alpha) Ie(\beta/\alpha, \alpha)$$

Thus

$$I_{a=0} = \pi e^{c/2} I_0(c/2) + (\pi b^2/\alpha) e^{b^2+c}\, Ie\left(\frac{c}{2\alpha}, \alpha\right) \tag{A3-6}$$

where

$$\alpha = b^2 + c/2 \tag{A3-7}$$