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Log-normal distribution

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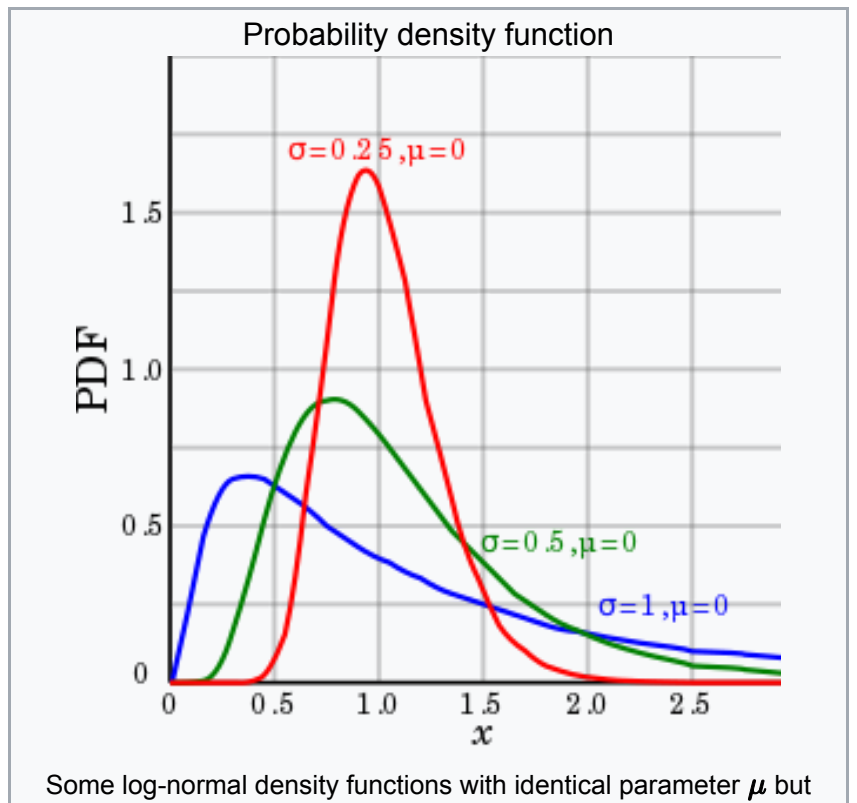
In probability theory, a **log-normal (or lognormal) distribution** is a continuous probability distribution of a random variable whose logarithm is normally distributed. Thus, if the random variable X is log-normally distributed, then $Y = \ln(X)$ has a normal distribution. Likewise, if Y has a normal distribution, then the exponential function of Y, $X = \exp(Y)$, has a log-normal distribution. A random variable which is log-normally distributed takes only positive real values. The distribution is occasionally referred to as the **Galton distribution** or **Galton's distribution**, after Francis Galton.[1] The log-normal distribution also has been associated with other names, such as McAlister, Gibrat and Cobb–Douglas.[1]

A log-normal process is the statistical realization of the multiplicative product of many independent random variables, each of which is positive. This is justified by considering the central limit theorem in the log domain. The log-normal distribution is the maximum entropy probability distribution for a random variate X for which the mean and variance of $\ln(X)$ are specified.[2]

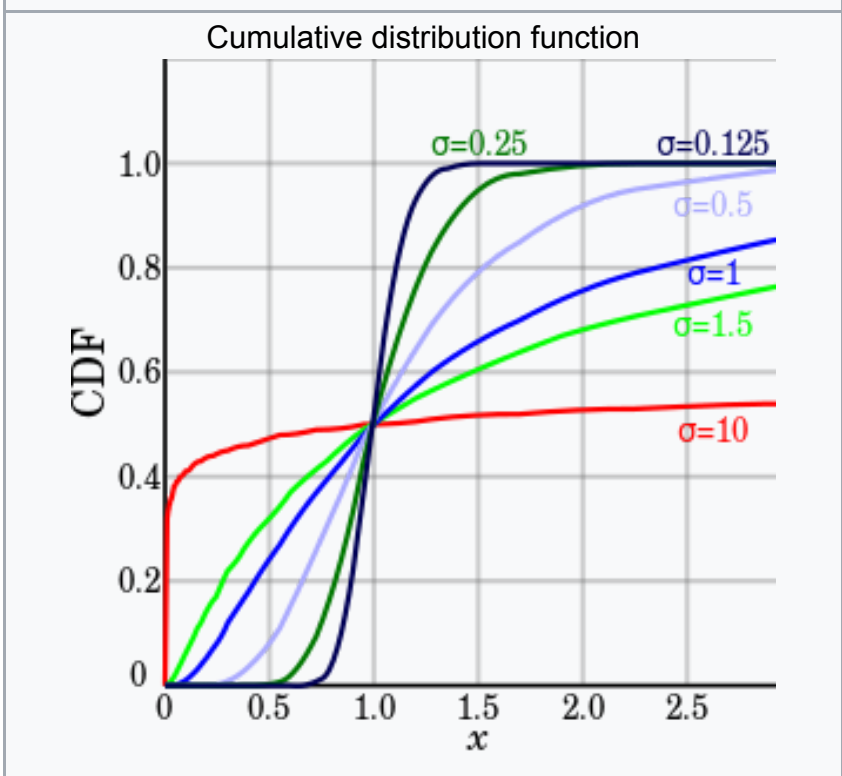
Log-normal

Probability density function



Some log-normal density functions with identical parameter μ but differing parameters σ

Cumulative distribution function



Cumulative distribution function of the log-normal distribution (with $\mu = 0$)

Notation	$\operatorname{Lognormal}(\mu, \sigma^2)$
Parameters	$\mu \in (-\infty, +\infty)$, $\sigma > 0$
Support	$x \in (0, +\infty)$
PDF	$\frac{1}{x\sigma\sqrt{2\pi}}\ e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}$
CDF	$\frac12 + \frac12\,\mathrm{erf}\Big[\frac{\ln x-\mu}{\sqrt{2}\sigma}\Big]$
Mean	$\exp\left(\mu + \frac{\sigma^2}{2}\right)$
Median	$\exp(\mu)$
Mode	$\exp(\mu - \sigma^2)$
Variance	$[\exp(\sigma^2) - 1] \exp(2\mu + \sigma^2)$
Skewness	$(e^{\sigma^2}\!\!+2) \sqrt{e^{\sigma^2}\!\!-1}$
Ex. kurtosis	$\exp(4\sigma^2) + 2\exp(3\sigma^2) + 3\exp(2\sigma^2) - 6$
Entropy	$\log(\sigma e^{\mu+\frac{1}{2}}\sqrt{2\pi})$
MGF	defined only for numbers with a non-positive real part, see text
CF	representation $\sum_{n=0}^{\infty}\frac{(it)^n}{n!}e^{n\mu+n^2\sigma^2/2}$ is asymptotically divergent but sufficient for numerical purposes
Fisher information	$\begin{pmatrix}1/\sigma^2&0\\0&1/2\sigma^4\end{pmatrix}$

Contents [hide]

Notation [edit]

Given a log-normally distributed random variable X and two parameters μ and σ that are, respectively, the mean and standard deviation of the variable's natural logarithm, then the logarithm of X is normally distributed, and we can write X as

$$X = e^{\mu+\sigma Z}$$

with Z a standard normal variable.

This relationship is true regardless of the base of the logarithmic or exponential function. If $\log_a(Y)$ is normally distributed, then so is $\log_b(Y)$, for any two positive numbers $a, b \neq 1$. Likewise, if e^X is log-normally distributed, then so is a^X, where a is a positive number $\neq 1$.

The two parameters μ and σ are not location and scale parameters for a lognormally distributed random variable *X*, but they are respectively location and scale parameters for the normally distributed logarithm **ln**(*X*). The quantity e^μ is a scale parameter for the family of lognormal distributions.

In contrast, the mean, standard deviation, and variance of the non-logarithmized sample values are respectively denoted m, *s.d.*, and v in this article. The two sets of parameters can be related as (see also Arithmetic moments below)[3]

$$\mu = \ln\left(\frac{m}{\sqrt{1+\frac{v}{m^2}}}\right), \qquad \sigma^2 = \ln\left(1+\frac{v}{m^2}\right).$$

Characterization [edit]

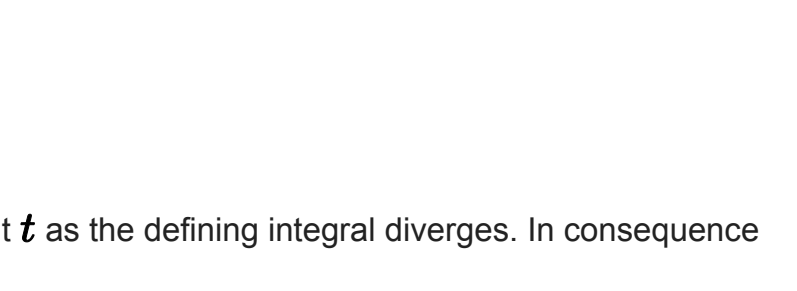
Probability density function [edit]

A positive random variable *X* is log-normally distributed if the logarithm of *X* is normally distributed,

$$\ln(X) \sim \mathcal{N}(\mu, \sigma^2).$$

Let Φ and φ be respectively the cumulative probability distribution function and the probability density function of the *N*(0,1) distribution.

Then we have[1]



Relation between normal and lognormal distribution. If $Y = \mu + \sigma Z$ is normally distributed, then $X \sim e^Y$ is lognormally distributed.

$$\begin{aligned} f_X(x) &= \frac{\mathrm d}{\mathrm dx}\Pr(X \le x) = \frac{\mathrm d}{\mathrm dx}\Pr(\ln X \le \ln x)\\ &= \frac{\mathrm d}{\mathrm dx}\Phi\left(\frac{\ln x-\mu}{\sigma}\right)\\ &= \varphi\left(\frac{\ln x-\mu}{\sigma}\right)\frac{\mathrm d}{\mathrm dx}\left(\frac{\ln x-\mu}{\sigma}\right)\\ &= \varphi\left(\frac{\ln x-\mu}{\sigma}\right)\frac{1}{\sigma x}\\ &= \frac{1}{x}\cdot\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right). \end{aligned}$$

Cumulative distribution function [edit]

The cumulative distribution function is

$$F_X(x) = \Phi\left(\frac{(\ln x)-\mu}{\sigma}\right)$$

where Φ is the cumulative distribution function of the standard normal distribution (i.e. *N*(0,1)).

This may also be expressed as follows:

$$\frac12\left[1+\operatorname{erf}\left(\frac{\ln x-\mu}{\sigma\sqrt2}\right)\right] = \frac12\operatorname{erfc}\left(-\frac{\ln x-\mu}{\sigma\sqrt2}\right)$$

where erfc is the complementary error function.

Characteristic function and moment generating function [edit]

All moments of the log-normal distribution exist and

$$\operatorname{E}[X^n] = e^{n\mu+n^2\sigma^2/2}$$

This can be derived by letting $z = \frac{\ln(x)-(\mu+n\sigma^2)}{\sigma}$ within the integral. However, the expected value $\operatorname{E}[e^{tX}]$ is not defined for any positive value of the argument t as the defining integral diverges. In consequence the moment generating function is not defined.[4] The last is related to the fact that the lognormal distribution is not uniquely determined by its moments.

The characteristic function $\operatorname{E}[e^{itX}]$ is defined for real values of *t* but is not defined for any complex value of *t* that has a negative imaginary part, and therefore the characteristic function is not analytic at the origin. In consequence, the characteristic function of the log-normal distribution cannot be represented as an infinite convergent series.[5] In particular, its Taylor formal series diverges:

$$\sum_{n=0}^\infty \frac{(it)^n}{n!}e^{n\mu+n^2\sigma^2/2}$$

However, a number of alternative divergent series representations have been obtained[5][6][7][8]

A closed-form formula for the characteristic function $\varphi(t)$ with t in the domain of convergence is not known. A relatively simple approximating formula is available in closed form and given by[9]

$$\varphi(t) \approx \frac{\exp\left(-\frac{W^2(t\sigma^2e^\mu)+2W(t\sigma^2e^\mu)}{2\sigma^2}\right)}{\sqrt{1+W(t\sigma^2e^\mu)}}$$

where W is the Lambert W function. This approximation is derived via an asymptotic method but it stays sharp all over the domain of convergence of φ.

Properties [edit]

Let $\operatorname{GM}[X]$ denote the geometric mean, and $\operatorname{GSD}[X]$ the geometric standard deviation of the random variable X, and let $\operatorname{E}[X]$ and $\operatorname{SD}[X]$ be the arithmetic mean, or expected value, and the arithmetic standard deviation as usual.

Geometric moments [edit]

The geometric mean of the log-normal distribution is $\operatorname{GM}[X] = e^\mu$, and the geometric standard deviation is $\operatorname{GSD}[X] = e^\sigma$.[10][11] By analogy with the arithmetic statistics, one can define a geometric variance, $\operatorname{GVar}[X] = e^{\sigma^2}$, and a geometric coefficient of variation,[10] $\operatorname{GCV}[X] = e^\sigma - 1$.

Because the log-transformed variable $Y = \ln X$ is symmetric and quantiles are preserved under monotonic transformations, the geometric mean of a log-normal distribution is equal to its median, $\operatorname{Med}[X]$.[12]

Note that the geometric mean is *less* than the arithmetic mean. This is due to the AM–GM inequality, and corresponds to the logarithm being convex down. In fact,

$$\operatorname{E}[X] = e^{\mu+\frac12\sigma^2} = e^\mu\cdot\sqrt{e^{\sigma^2}} = \operatorname{GM}[X]\cdot\sqrt{\operatorname{GVar}[X]}.$$

In finance the term $e^{-\frac12\sigma^2}$ is sometimes interpreted as a convexity correction. From the point of view of stochastic calculus, this is the same correction term as in Itō's lemma for geometric Brownian motion.

Arithmetic moments [edit]

For any real or complex number n, the n-th moment of a log-normally distributed variable X is given by[1]

$$\operatorname{E}[X^n] = e^{n\mu+\frac12n^2\sigma^2}.$$

Specifically, the arithmetic mean, expected square, arithmetic variance, and arithmetic standard deviation of a log-normally distributed variable X are given by

$$\begin{aligned} &\operatorname{E}[X] = e^{\mu+\frac12\sigma^2},\\ &\operatorname{E}[X^2] = e^{2\mu+2\sigma^2},\\ &\operatorname{Var}[X] = \operatorname{E}[X^2]-[EX]^2 = (\operatorname{E}[X])^2(e^{\sigma^2}-1) = e^{2\mu+\sigma^2}(e^{\sigma^2}-1),\\ &\operatorname{SD}[X] = \sqrt{\operatorname{Var}[X]} = \operatorname{E}[X]\sqrt{e^{\sigma^2}-1} = e^{\mu+\frac12\sigma^2}\sqrt{e^{\sigma^2}-1}, \end{aligned}$$

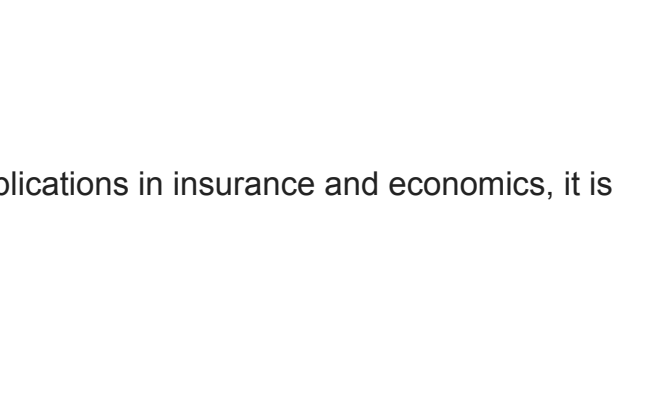
respectively.

The parameters μ and σ can be obtained if the arithmetic mean and the arithmetic variance are known:

$$\begin{aligned} \mu &= \ln\left(\frac{\operatorname{E}[X]^2}{\sqrt{\operatorname{E}[X^2]}}\right) = \ln\left(\frac{\operatorname{E}[X]^2}{\sqrt{\operatorname{Var}[X]+\operatorname{E}[X]^2}}\right),\\ \sigma^2 &= \ln\left(\frac{\operatorname{E}[X^2]}{\operatorname{E}[X]^2}\right) = \ln\left(1+\frac{\operatorname{Var}[X]}{\operatorname{E}[X]^2}\right). \end{aligned}$$

A probability distribution is not uniquely determined by the moments $\operatorname{E}[X^n] = e^{n\mu+\frac12n^2\sigma^2}$ for $n \geq 1$. That is, there exist other distributions with the same set of moments.[1] In fact, there is a whole family of distributions with the same moments as the log-normal distribution.[*citation needed*]

Mode and median [edit]



Comparison of mean, median and mode of two log-normal distributions with different skewness.

The mode is the point of global maximum of the probability density function. In particular, it solves the equation $(\ln f)' = 0$:

$$\operatorname{Mode}[X] = e^{\mu-\sigma^2}.$$

The median is such a point where $F_X = 0.5$:

$$\operatorname{Med}[X] = e^\mu.$$

Arithmetic coefficient of variation [edit]

The arithmetic coefficient of variation $\operatorname{CV}[X]$ is the ratio $\frac{\operatorname{SD}[X]}{\operatorname{E}[X]}$ (on the natural scale). For a log-normal distribution it is equal to

$$\operatorname{CV}[X] = \sqrt{e^{\sigma^2}-1}.$$

Contrary to the arithmetic standard deviation, the arithmetic coefficient of variation is independent of the arithmetic mean.

Partial expectation [edit]

The partial expectation of a random variable X with respect to a threshold k is defined as

$$g(k) = \int_k^\infty x f_X(x)\,dx.$$

Alternatively, and using the definition of conditional expectation, it can be written as $g(k) = \operatorname{E}[X \mid X>k]P(X>k)$. For a log-normal random variable the partial expectation is given by:

$$g(k) = \int_k^\infty x f_X(x)\,dx = e^{\mu+\frac12\sigma^2}\,\Phi\left(\frac{\mu+\sigma^2-\ln k}{\sigma}\right)$$

where Φ is the normal cumulative distribution function. The derivation of the formula is provided in the discussion of this Wikipedia entry. The partial expectation formula has applications in insurance and economics, it is used in solving the partial differential equation leading to the Black–Scholes formula.

Conditional expectation [edit]

The conditional expectation of a lognormal random variable X with respect to a threshold k is its partial expectation divided by the cumulative probability of being in that range:

$$E[X \mid X<k] = e^{\mu+\frac{\sigma^2}{2}}\cdot\frac{\Phi\left[\frac{\ln(k)-\mu-\sigma^2}{\sigma}\right]}{\Phi\left[\frac{\ln(k)-\mu}{\sigma}\right]}$$

$$E[X \mid X\geqslant k] = e^{\mu+\frac{\sigma^2}{2}}\cdot\frac{\Phi\left[\frac{\mu+\sigma^2-\ln(k)}{\sigma}\right]}{1-\Phi\left[\frac{\ln(k)-\mu}{\sigma}\right]}$$

Other [edit]

A set of data that arises from the log-normal distribution has a symmetric Lorenz curve (see also Lorenz asymmetry coefficient).[13]

The harmonic H, geometric G and arithmetic A means of this distribution are related;[14] such relation is given by

$$H = \frac{G^2}{A}.$$

Log-normal distributions are infinitely divisible,[15] but they are not stable distributions, which can be easily drawn from.[16]

Occurrence and applications [edit]

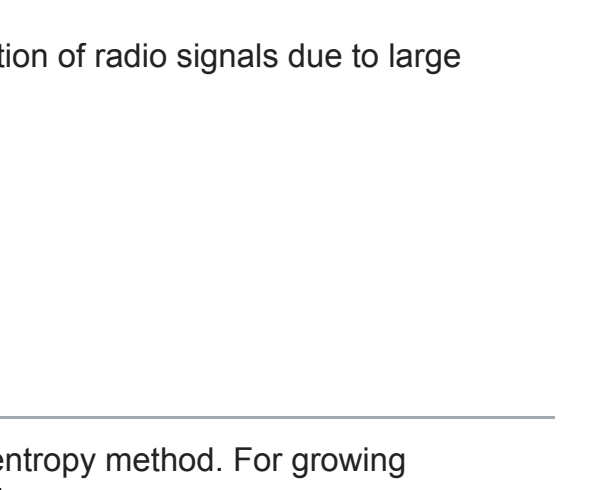
The log-normal distribution is important in the description of natural phenomena. This follows, because many natural growth processes are driven by the accumulation of many small percentage changes. These become additive on a log scale. If the effect of any one change is negligible, the central limit theorem says that the distribution of their sum is more nearly normal than that of the summands. When back-transformed onto the original scale, it makes the distribution of sizes approximately log-normal (though if the standard deviation is sufficiently small, the normal distribution can be an adequate approximation).

This multiplicative version of the central limit theorem is also known as Gibrat's law, after Robert Gibrat (1904–1980) who formulated it for companies.[17] If the rate of accumulation of these small changes does not vary over time, growth becomes independent of size. Even if that's not true, the size distributions at any age of things that grow over time tends to be log-normal.

Examples include the following:

- Human behaviors
  - The length of comments posted in Internet discussion forums follows a log-normal distribution.[18]
  - Users' dwell time on online articles (jokes, news etc.) follows a log-normal distribution.[19]
  - Onset durations of acoustic comparison stimuli that are matched to a standard stimulus follow a log-normal distribution.[20]
- In biology and medicine,
  - Measures of size of living tissue (length, skin area, weight);[21]
  - For highly communicable epidemics, such as SARS in 2003, if publication intervention is involved, the number of hospitalized cases is shown to satisfy the lognormal distribution with no free parameters if an entropy is assumed and the standard deviation is determined by the principle of maximum rate of entropy production.[22]
  - The length of inert appendages (hair, claws, nails, teeth) of biological specimens, in the direction of growth;[*citation needed*]
  - The normalised RNA-Seq readcount for any genomic region can be well approximated by log-normal distribution.
  - Certain physiological measurements, such as blood pressure of adult humans (after separation on male/female subpopulations)[23]
  - In neuroscience, the distribution of firing rates across a population of neurons is often approximately lognormal. This has been first observed in the cortex and striatum [24] and later in hippocampus and entorhinal cortex,[25] and elsewhere in the brain.[26][27] Also, intrinsic gain distributions and synaptic weight distributions appear to be lognormal[28] as well.
- In colloidal chemistry and polymer chemistry
  - Particle size distributions
  - Molar mass distributions

Consequently, reference ranges for measurements in healthy individuals are more accurately estimated by assuming a log-normal distribution than by assuming a symmetric distribution about the mean.



Fitted cumulative log-normal distribution to annually maximum 1-day rainfalls, see distribution fitting

- In hydrology, the log-normal distribution is used to analyze extreme values of such variables as monthly and annual maximum values of daily rainfall and river discharge volumes.[29]

  The image on the right, made with CumFreq, illustrates an example of fitting the log-normal distribution to ranked annually maximum one-day rainfalls showing also the 90% confidence belt based on the binomial distribution.[30]

  The rainfall data are represented by plotting positions as part of a cumulative frequency analysis.

- In social sciences and demographics
  - In economics, there is evidence that the income of 97%–99% of the population is distributed log-normally.[31] (The distribution of higher-income individuals follows a Pareto distribution.[32])
  - In finance, in particular the Black–Scholes model, changes in the *logarithm* of exchange rates, price indices, and stock market indices are assumed normal[33] (these variables behave like compound interest, not like simple interest, and so are multiplicative). However, some mathematicians such as Benoît Mandelbrot have argued [34] that log-Lévy distributions, which possesses heavy tails would be a more appropriate model, in particular for the analysis for stock market crashes. Indeed, stock price distributions typically exhibit a fat tail.;[35] the fat tailed distribution of changes during stock market crashes invalidate the assumptions of the central limit theorem.
  - City sizes.
- Technology
  - In reliability analysis, the lognormal distribution is often used to model times to repair a maintainable system.[36]
  - In wireless communication, "the local-mean power expressed in logarithmic values, such as dB or neper, has a normal (i.e., Gaussian) distribution." [37] Also, the random obstruction of radio signals due to large buildings and hills, called shadowing, is often modeled as a lognormal distribution.
  - Particle size distributions produced by comminution with random impacts, such as in ball milling
  - The file size distribution of publicly available audio and video data files (MIME types) follows a log-normal distribution over five orders of magnitude.[38]
- The length of chess games tends to follow a log normal distribution.[39]

Extremal principle of entropy to fix the free parameter σ [edit]

- In applications, σ is a parameter to be determined. In cases that there are no data to determine this parameter, it is possible to evaluate it from some universal principle. One is the entropy method. For growing processes which are governed by production and dissipation, it was shown that one can use some extremal principle of Shannon entropy to determine this parameter to be $\sigma = 1/\sqrt{6}$. This value can then be used to give some scaling relation between the inflexion point and maximum point of the lognormal distribution.[40] It is shown that this relationship is determined by the base of natural logarithm, $e = 2.718\ldots$, and exhibits some geometrical similarity to the minimal surface energy principle. These scaling relations are shown to be useful for predicting a number of growth processes (epidemic spreading, droplet splashing, population growth, swirling rate of the bathtub vortex, distribution of language characters, velocity profile of turbulences, etc.). For instance, the lognormal function with such σ fits well with the size of secondary produced droplet during droplet impact [41] and the spreading of one epidemic disease.[22]

Maximum likelihood estimation of parameters [edit]

For determining the maximum likelihood estimators of the log-normal distribution parameters *μ* and *σ*, we can use the same procedure as for the normal distribution. To avoid repetition, we observe that

$$L(\mu,\sigma) = \prod_{i=1}^n \frac{1}{x_i}\varphi_{\mu,\sigma}(\ln x_i)$$

where φ is the density function of the normal distribution $\mathcal{N}(\mu,\sigma^2)$. Therefore, using the same indices to denote distributions, we can write the log-likelihood function thus:

$$\begin{aligned} \ell(\mu,\sigma \mid x_1,x_2,\ldots,x_n) &= -\sum_k \ln x_k + \ell_N(\mu,\sigma \mid \ln x_1,\ln x_2,\ldots,\ln x_n)\\ &= \text{constant} + \ell_N(\mu,\sigma \mid \ln x_1,\ln x_2,\ldots,\ln x_n). \end{aligned}$$

Since the first term is constant with regard to *μ* and *σ*, both logarithmic likelihood functions, ℓ_L and ℓ_N, reach their maximum with the same μ and σ. Hence, using the formulas for the normal distribution maximum likelihood parameter estimators and the equality above, we deduce that for the log-normal distribution it holds that

$$\widehat\mu = \frac{\sum_k \ln x_k}{n}, \qquad \widehat{\sigma}^2 = \frac{\sum_k\left(\ln x_k-\widehat\mu\right)^2}{n}.$$

Multivariate log-normal [edit]

If $\boldsymbol X \sim \mathcal{N}(\boldsymbol\mu,\,\boldsymbol\Sigma)$ is a multivariate normal distribution then $\boldsymbol Y = \exp(\boldsymbol X)$ has a multivariate log-normal distribution[42][43] with mean

$$\operatorname{E}[\boldsymbol Y]_i = e^{\mu_i+\frac12\Sigma_{ii}},$$

and covariance matrix

$$\operatorname{Var}[\boldsymbol Y]_{ij} = e^{\mu_i+\mu_j+\frac12(\Sigma_{ii}+\Sigma_{jj})}(e^{\Sigma_{ij}}-1).$$

Related distributions [edit]

- If $X \sim \mathcal{N}(\mu,\sigma^2)$ is a normal distribution, then $\exp(X) \sim \operatorname{Lognormal}(\mu,\sigma^2)$.
- If $X \sim \operatorname{Lognormal}(\mu,\sigma^2)$ is distributed log-normally, then $\ln(X) \sim \mathcal{N}(\mu,\sigma^2)$ is a normal random variable.
- If $X_j \sim \operatorname{Lognormal}(\mu_j,\sigma_j^2)$ are n independent log-normally distributed variables, and $Y = \prod_{j=1}^n X_j$, then Y is also distributed log-normally:

  $$Y \sim \operatorname{Lognormal}\left(\textstyle\sum_{j=1}^n\mu_j,\ \sum_{j=1}^n\sigma_j^2\right).$$

- Let $X_j \sim \operatorname{Lognormal}(\mu_j,\sigma_j^2)$ be independent log-normally distributed variables with possibly varying σ and μ parameters, and $Y = \sum_{j=1}^n X_j$. The distribution of Y has no closed-form expression, but can be reasonably approximated by another log-normal distribution Z at the right tail.[44] Its probability density function at the neighborhood of 0 has been characterized[16] and it does not resemble any log-normal distribution. A commonly used approximation due to L.F. Fenton (but previously stated by R.I. Wilkinson and mathematical justified by Marlow[45]) is obtained by matching the mean and variance of another lognormal distribution:

  $$\begin{aligned} \sigma^2_Z &= \ln\left[\frac{\sum e^{2\mu_j+\sigma_j^2}(e^{\sigma_j^2}-1)}{(\sum e^{\mu_j+\sigma_j^2/2})^2}+1\right],\\ \mu_Z &= \ln\left[\sum e^{\mu_j+\sigma_j^2/2}\right]-\frac{\sigma^2_Z}{2}. \end{aligned}$$

  In the case that all X_j have the same variance parameter $\sigma_j = \sigma$, these formulas simplify to

  $$\begin{aligned} \sigma^2_Z &= \ln\left[(e^{\sigma^2}-1)\frac{\sum e^{2\mu_j}}{(\sum e^{\mu_j})^2}+1\right],\\ \mu_Z &= \ln\left[\sum e^{\mu_j}\right]+\frac{\sigma^2}{2}-\frac{\sigma^2_Z}{2}. \end{aligned}$$

- If $X \sim \operatorname{Lognormal}(\mu,\sigma^2)$ then $X+c$ is said to have a *shifted log-normal* distribution with support $x \in (c,+\infty)$. $\operatorname{E}[X+c] = \operatorname{E}[X]+c$, $\operatorname{Var}[X+c] = \operatorname{Var}[X]$.
- If $X \sim \operatorname{Lognormal}(\mu,\sigma^2)$ then $aX \sim \operatorname{Lognormal}(\mu+\ln a,\ \sigma^2)$.
- If $X \sim \operatorname{Lognormal}(\mu,\sigma^2)$ then $\frac{1}{X} \sim \operatorname{Lognormal}(-\mu,\ \sigma^2)$.
- If $X \sim \operatorname{Lognormal}(\mu,\sigma^2)$ then $X^a \sim \operatorname{Lognormal}(a\mu,\ a^2\sigma^2)$ for $a \neq 0$.
- Lognormal distribution is a special case of semi-bounded Johnson distribution
- If $X \mid Y \sim \operatorname{Rayleigh}(Y)$ with $Y \sim \operatorname{Lognormal}(\mu,\sigma^2)$, then $X \sim \operatorname{Suzuki}(\mu,\sigma)$ (Suzuki distribution)
- A substitute for the log-normal whose integral can be expressed in terms of more elementary functions[46] can be obtained based on the logistic distribution to get an approximation for the CDF

  $$F(x;\mu,\sigma) = \left[\left(\frac{e^\mu}{x}\right)^{\pi/(\sigma\sqrt3)}+1\right]^{-1}.$$

  This is a log-logistic distribution.

See also [edit]

- Log-distance path loss model
- Slow fading

Notes [edit]

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